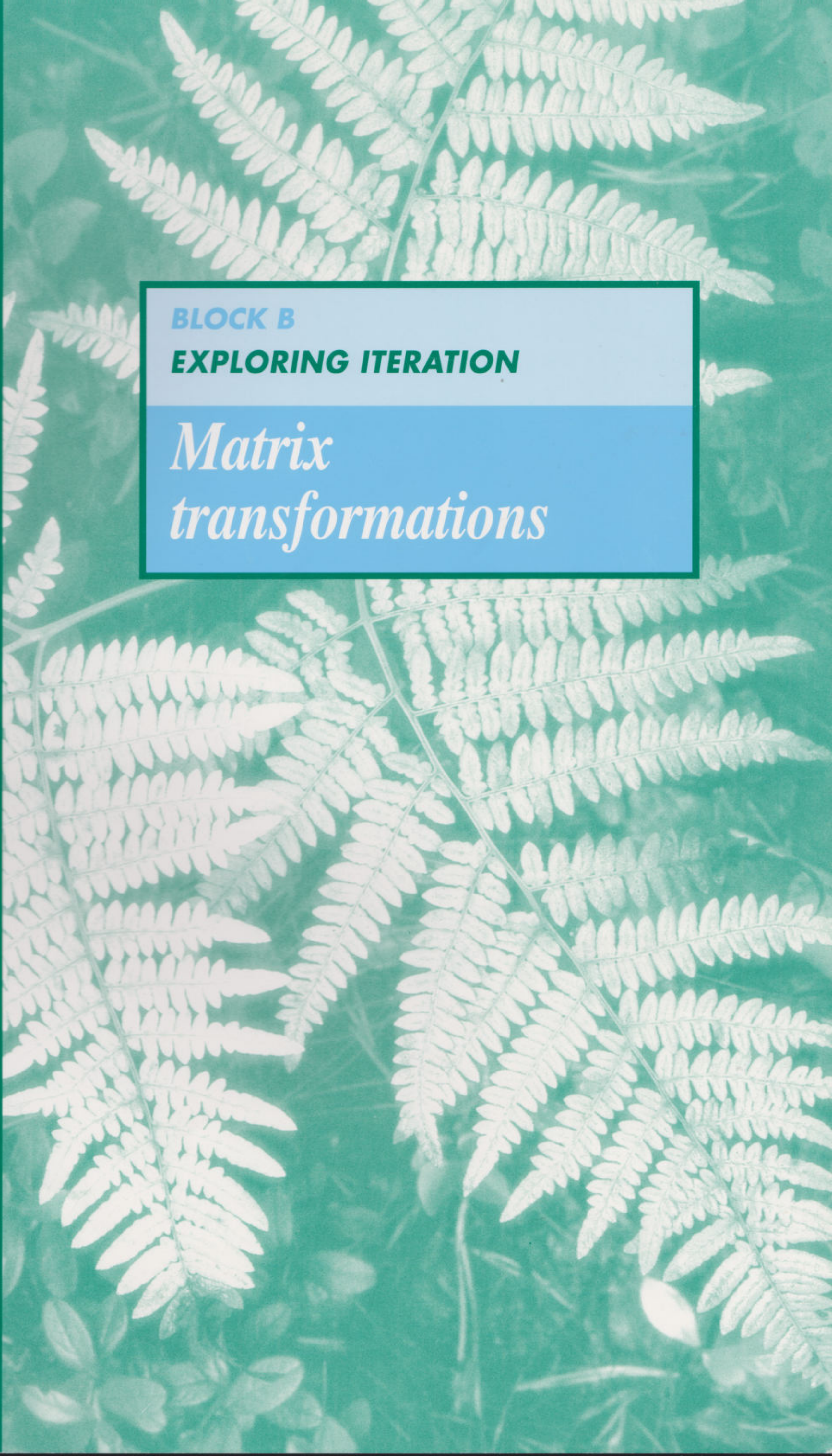
MS221 Chapter B2

The Open University

A second level interdisciplinary course

Exploring Mathematics



BLOCK B

EXPLORING ITERATION

Matrix transformations

CHAPTER

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Prepared by the course team

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This course, MS221 *Exploring Mathematics*, and the courses MU120 *Open Mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

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Contents

Study guide

There are five sections to this chapter. They are intended to be studied consecutively in five study sessions. Section 5 requires the use of the computer and Computer Book B.

The pattern of study for each session might be as follows.

Study session 1: Section 1.

Study session 2: Section 2.

Study session 3: Section 3.

Study session 4: Section 4.

Study session 5: Section 5.

Each session requires two to three hours, the longest being the second and the shortest being the fourth.

Before studying this chapter, you should be familiar with the following topics:

◇ addition and scalar multiplication of vectors, including the geometric representation of these operations;

◇ addition and multiplication of 2×2 matrices;

◇ the properties

$$\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{A}\mathbf{u}+\mathbf{A}\mathbf{v},$$
$$(\mathbf{A}\mathbf{B})\mathbf{u} = \mathbf{A}(\mathbf{B}\mathbf{u}),$$
$$\mathbf{A}(k\mathbf{u}) = k(\mathbf{A}\mathbf{u}),$$

where $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices, $\mathbf{u}$ and $\mathbf{v}$ are vectors with two components, and k is a scalar;

◇ the inverse of a 2×2 matrix that has non-zero determinant.

> The optional Video Band B(ii) *Algebra workout – Finding matrix transformations* could be viewed at any stage during your study of this chapter.

Introduction

In Chapter A3 you saw how functions, known as *isometries*, can be used to move figures around the plane without altering their size or shape. In this chapter we explore a more general class of functions, known as *affine transformations*, which need not preserve size and shape but which do preserve 'parallelism'; that is, they map parallel lines to parallel lines. These are the kinds of transformation found in many computer graphics packages for manipulating selected polygonal figures on the screen. They include translations, rotations, reflections, scalings and shears, such as those illustrated in Figure 0.1.

Since the functions we deal with in this chapter are geometric in nature, they will usually be referred to as 'transformations' (see Chapter A3, page 6).

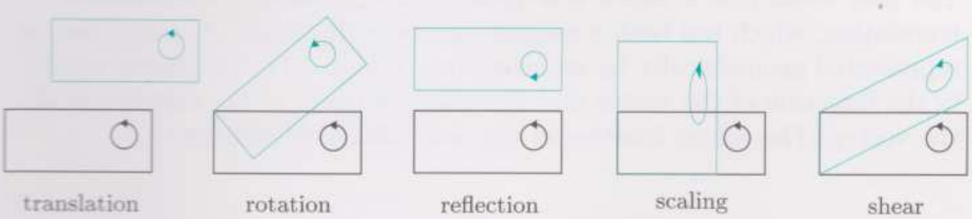


Figure 0.1 Some affine transformations

In all these examples, apart from the reflection, the orientation of the image is the same as that of the original. Orientation is indicated by the directed 'circles' on the rectangles and their images.

In Chapter A3, the rules for isometries were expressed in terms of the Cartesian coordinates of points in the plane. This chapter adopts a slightly different description of points, in terms of vectors, and this enables us to express the rules for certain isometries in a convenient matrix form, namely: $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix and $\mathbf{x}$ is a vector. By considering other functions that have the same matrix form, we are led to define *linear transformations* of the plane. If a linear transformation f has a rule of the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, then the matrix $\mathbf{A}$ is said to *represent* f. Linear transformations have the property that they can be composed or inverted simply by multiplying or inverting the matrices that represent them.

The linear transformations include all those transformations that fix the origin and preserve parallelism. Unfortunately, they exclude many other transformations, like translations, that also preserve parallelism. By combining linear transformations with translations, we arrive at *affine transformations*, which have rules of the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$, where $\mathbf{a}$ is a vector.

Section 1 introduces a representation of points in the plane in terms of vectors and uses them to express isometries in terms of matrices.

Section 2 generalises the matrix representations of isometries by introducing *linear transformations*. The geometric behaviour of some linear transformations is described, including *scalings* and *shears*. Determinants are then used as a way of calculating the effect that linear transformations have on areas. Section 3 shows how *composite* linear transformations (the result of following one linear transformation by another) and *inverse* linear transformations (the result of undoing a linear transformation) can be expressed in terms of matrices.

Section 4 defines affine transformations, and shows how they can be used to describe arbitrary rotations and reflections of the plane.

In Section 5, the computer is used to explore the various types of linear and affine transformations.

1 Vectors and matrices

In Chapter A3, you saw how to represent isometries by functions that map the set $\mathbb{R}^2$ to itself. This section describes how to express the rule for such functions in terms of matrices. Later you will see that this makes it easier to compose and invert isometries. First, you will see how vectors can be used to describe the position of points in the plane.

Vectors were discussed in MST121 Chapters B2 and B3.

1.1 Vectors

You may recall that a vector is a quantity, like velocity, acceleration or translation, which has both a magnitude and a direction. A vector can be represented geometrically by an arrow (see Figure 1.1). The arrow points in the direction of the vector and has a length equal to the magnitude of the vector. The actual location of the arrow does not matter.

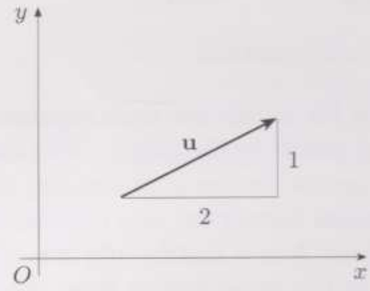


Figure 1.1 Geometrical representation of a vector and its components

For calculations it is often more useful to represent the vector by a column matrix that lists the x- and y-components of the vector. The x-component is the horizontal displacement from the tail of the arrow to the tip, and the y-component is the vertical displacement from the tail to the tip. For example, the vector in Figure 1.1 can be written as $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where 2 is the x-component and 1 is the y-component.

Here *horizontal* means parallel to the x-axis and *vertical* means parallel to the y-axis.

Often a vector is denoted by a single letter such as $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{u}$. In print, it is conventional to typeset the letter in bold to distinguish it from other quantities. In handwriting, the symbol for a vector is usually underlined with a straight (or wavy) line.

Do remember to underline vectors in your written work.

Activity 1.1 Sketching vectors with given components

Draw an arrow on the (x, y)-plane that represents each of the following vectors. In each case, mark the components on your diagram.

(a) $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (b) $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (c) $\mathbf{c} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ (d) $\mathbf{d} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

Solutions are given on page 55.

We perform the operations of addition, subtraction, taking the negative, and scalar multiplication on vectors by performing the corresponding operations on the components of the vectors. For example, if, as in Activity 1.1, $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then

$$2\mathbf{a} - \mathbf{b} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

These operations were discussed in MST121 Chapters B2 and B3.

Activity 1.2 Combining vectors algebraically

Let $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$. Calculate each of the following vectors $\mathbf{c}$.

(a) $\mathbf{c} = 4\mathbf{a}$ (b) $\mathbf{c} = -2\mathbf{b}$ (c) $\mathbf{c} = 4\mathbf{a} - 2\mathbf{b}$ (d) $\mathbf{c} = 2\mathbf{a} + \frac{3}{2}\mathbf{b}$

Solutions are given on page 55.

The vector operations of addition, subtraction, taking the negative and scalar multiplication all have convenient geometric interpretations. The following examples are shown in Figure 1.2.

- $2\mathbf{a}$ is the vector in the same direction as $\mathbf{a}$, but with twice the magnitude.
- $-\mathbf{b}$ is the vector with the same magnitude as $\mathbf{b}$, but the opposite direction.
- $2\mathbf{a} - \mathbf{b} = 2\mathbf{a} + (-\mathbf{b})$ is the vector obtained by adding $2\mathbf{a}$ to $-\mathbf{b}$, which is achieved as follows. If the tail of the arrow for $-\mathbf{b}$ is placed at the tip of the arrow for $2\mathbf{a}$, then the arrow from the tail of the arrow for $2\mathbf{a}$ to the tip of the arrow for $-\mathbf{b}$ is the arrow for $2\mathbf{a} + (-\mathbf{b})$.

This process, known as the *triangle rule for addition of vectors*, was discussed in MST121 Chapter B3, Section 1.

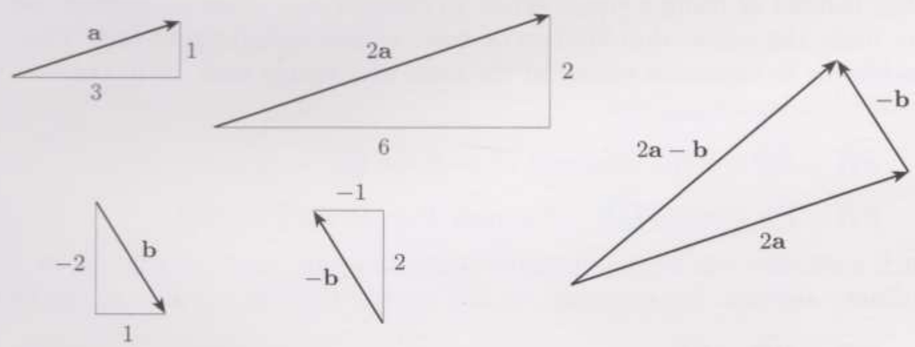


Figure 1.2 Combining vectors geometrically

Activity 1.3 Combining vectors geometrically

For each vector $\mathbf{c}$ calculated in Activity 1.2, draw a sketch that indicates its relationship to $\mathbf{a}$ and $\mathbf{b}$.

Solutions are given on page 55.

Vectors are often used when working with geometric figures in which points are denoted by letters. In this situation, it is convenient to use an alternative 'arrow' notation. We write $\overrightarrow{PQ}$ to denote the (displacement) vector from P to Q, which is represented by the arrow that connects P to Q. Figure 1.3 shows a parallelogram $PQRS$ in which several such vectors have been introduced.

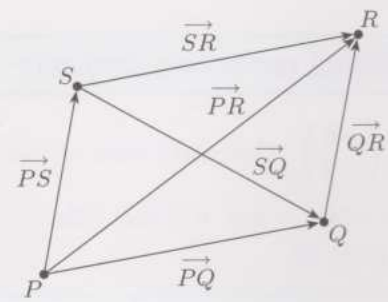


Figure 1.3 Arrow notation for vectors

Some arrows in the figure have the same length and direction, and therefore represent the same vector. For example,

$$\overrightarrow{PQ} = \overrightarrow{SR} \quad \text{and} \quad \overrightarrow{PS} = \overrightarrow{QR}.$$

To represent the negative of a vector we can change the order of the letters. For example, $\overrightarrow{QP}$ is the negative of the vector from P to Q; that is,

$$-\overrightarrow{PQ} = \overrightarrow{QP}.$$

Hence

$$\overrightarrow{QP} = -\overrightarrow{SR}.$$

Also, instead of using a single arrow to connect one point to another, we can make the connection via two or more arrows linked tip to tail. This enables us to express a vector in the form of a vector sum, as in the following vector equalities.

- $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$ (connect P to R via Q)
- $\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR}$ (connect P to R via S and Q)

Such equalities can be manipulated using the same kinds of rules as in ordinary algebra. For example, we can rewrite the first equality in the form

Such manipulations are valid because vector operations are defined by the effect that the corresponding real operations have on the components of the vectors, so the rules of algebra 'carry over' from the components to the vectors.

$$\overrightarrow{PQ} = \overrightarrow{PR} - \overrightarrow{QR}$$

by subtracting $\overrightarrow{QR}$ from both sides.

Activity 1.4 Manipulating vector equalities

This activity relates to Figure 1.3.

(a) Express $\overrightarrow{QR}$ in terms of $\overrightarrow{PR}$ and $\overrightarrow{SR}$.

(b) Express $\overrightarrow{PS}$ in terms of $\overrightarrow{PR}$ and $\overrightarrow{PQ}$.

Solutions are given on page 56.

1.2 Position vectors

In geometry we often use vectors to identify the position of points relative to the origin O. When used in this way, vectors are called *position vectors.*

> Let P be any point in the plane. Then the vector $\overrightarrow{OP}$ is called the **position vector of P (with respect to O)**. It is usually denoted by the corresponding lower-case letter $\mathbf{p}$ (or by $\underline{p}$ when handwritten); see Figure 1.4.

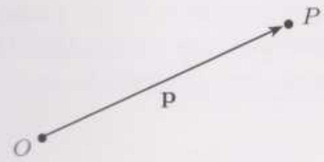


Figure 1.4 Point P with position vector $\mathbf{p}$

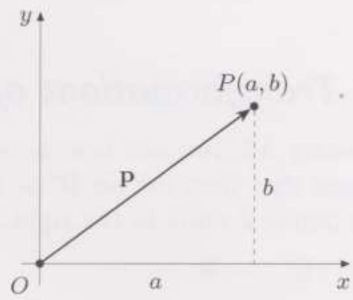


Figure 1.5 Coordinates and components

There is a close relationship between the coordinates of a point and the components of its position vector. For if P is a point with coordinates (a, b), then its position vector $\mathbf{p} = \overrightarrow{OP}$ is $\begin{pmatrix} a \\ b \end{pmatrix}$ (see Figure 1.5). In other words, the coordinates of the point P are equal to the components of its position vector $\mathbf{p}$. For this reason, we often refer to 'the point $\mathbf{p}$' when strictly speaking we should say 'the point P', or 'the point with position vector $\mathbf{p}$'. For the same reason, we denote the plane by $\mathbb{R}^2$ irrespective of whether its points are represented by coordinates or by position vectors.

As an illustration of how position vectors are used, suppose that you are given two points P and Q, with position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, and that you want to find the position vector $\mathbf{r}$ of the point R midway between P and Q, as shown in Figure 1.6.

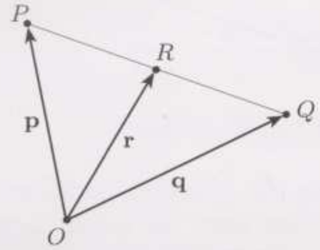


Figure 1.6 Locating R

Since R is the midpoint of PQ, it follows that $\overrightarrow{PR} = \frac{1}{2}\overrightarrow{PQ}$; so

$$\mathbf{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{PQ}.$$

But $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$, so

$$\mathbf{r} = \mathbf{p} + \tfrac{1}{2}(\mathbf{q} - \mathbf{p}) = \tfrac{1}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{q}.$$

For example, if $P = (4, 7)$ and $Q = (2, -3)$, then R has position vector

$$\tfrac{1}{2}\begin{pmatrix} 4 \\ 7 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{7}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Activity 1.5 Finding position vectors of points on a line

(a) Find the position vectors (in terms of $\mathbf{p}$ and $\mathbf{q}$) of the points S and T, where S lies a quarter of the way from P to Q, and T lies a third of the way from Q to P, as shown in Figure 1.7.

(b) Determine S and T in the particular case where $P = (2, 5)$ and $Q = (1, -3)$.

Solutions are given on page 56.

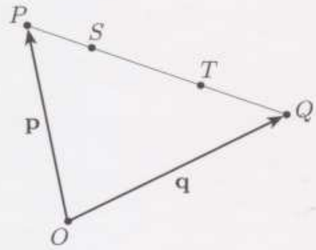


Figure 1.7 Locating S and T

1.3 Transformations of the plane

In Chapter A3, you saw how to represent transformations of the plane by functions that map the set $\mathbb{R}^2$ to itself. For example, the translation that moves points 2 units to the right, and 3 units up, can be written as

$$\begin{aligned} t_{2,3} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x + 2, y + 3). \end{aligned}$$

In terms of position vectors, the rule for this function becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+2 \\ y+3 \end{pmatrix}.$$

In effect, this rule adds the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ to $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, where $\mathbf{x}$ is the position vector of any point to which the transformation is applied. The effect of the translation is therefore to send $\mathbf{x}$ to $\mathbf{x} + \mathbf{a}$ (see Figure 1.8).

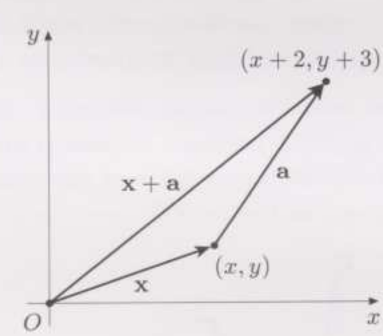


Figure 1.8 Translation through the vector $\mathbf{a}$

It follows from the triangle law of vector addition that $\mathbf{a}$ is the vector represented by the arrow that joins the point (x, y) to $(x + 2, y + 3)$. This in turn means that $\mathbf{a}$ has magnitude and direction equal to the distance and direction through which $t_{2,3}$ moves points. We therefore say that $t_{2,3}$ translates points (or figures) 'through' the vector $\mathbf{a}$.

Vector description of translation

A **translation** of the plane through the vector $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ has the form

$$\begin{aligned} t_{p,q} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{x} + \mathbf{a}. \end{aligned}$$

Rotations

The main advantage of switching to vector notation emerges when we consider other transformations of the plane such as rotations and reflections. In Chapter A3, you saw that a rotation about the origin through θ radians is represented by the function

$$\begin{aligned} r_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

If θ is positive, then this function represents an anticlockwise rotation; if θ is negative, then the rotation is clockwise, as shown in Figure 1.9.

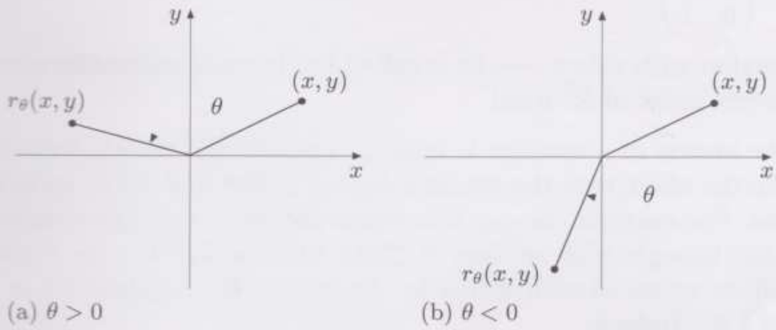


Figure 1.9 Rotation through θ

In vector form, the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ under the rotation r_θ is

$$\begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix},$$

which can be written as the matrix product

$$\begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In this sense, the rotation through θ about the origin is represented by the matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.1}$$

for if $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the image of $\mathbf{x}$ under the rotation is $\mathbf{R}_\theta\mathbf{x}$.

> **Matrix description of rotation**
>
> A **rotation** of the plane about the origin, through an angle θ, has the form
>
> $$\begin{aligned} r_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{R}_\theta\mathbf{x}, \end{aligned}$$
>
> where $\mathbf{R}_\theta$ is the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

From here on, since all angles are measured in radians, the unit will not usually be mentioned explicitly.

For example, the matrix $\mathbf{R}_\theta$ for an anticlockwise rotation through one quarter of a turn is obtained by setting $\theta = \pi/2$. Since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$, we obtain $\mathbf{R}_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Activity 1.6 Rotations in matrix form

Find the matrices which represent the rotations r_0, r_π, $r_{3\pi/2}$, $r_{2\pi}$.

Solutions are given on page 56.

The *identity* matrix was introduced in MST121 Chapter B2, Section 5.

In Activity 1.6 you saw that a rotation through 0 radians (or through 2π radians) is represented by the *identity* matrix $\mathbf{I}$, where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The function with rule $\mathbf{x} \longmapsto \mathbf{I}\mathbf{x}$ is called the **identity transformation**; it leaves every point of $\mathbb{R}^2$ fixed.

Every isometry preserves the shape and size of a figure; that is, a figure and its image are congruent. In particular, the image of a polygon is specified by the images of the vertices of the polygon.

Once the matrix of a rotation is known, it is a straightforward matter to calculate the effect that the rotation has on points and, more generally, on polygons. For example, we can determine the effect that the rotation $r_{3\pi/2}$ has on the triangle with vertices at $(2,1)$, $(3,0)$ and $(-1,1)$ by multiplying the position vector of each vertex by the matrix $\mathbf{R}_{3\pi/2}$ (obtained in Activity 1.6). Indeed:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix};$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix};$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So the image is the congruent triangle with vertices at $(1,-2)$, $(0,-3)$ and $(1,1)$, as shown in Figure 1.10.

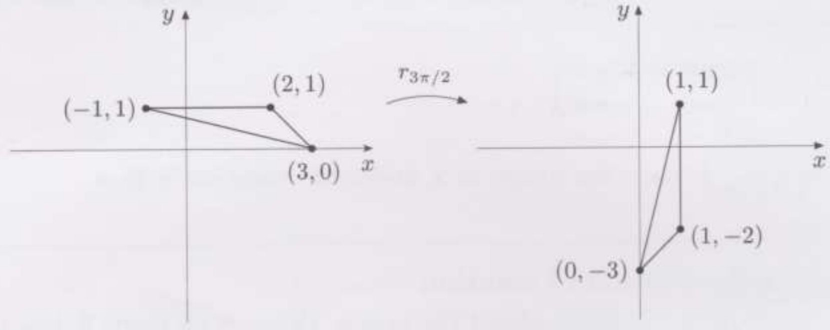


Figure 1.10 An anticlockwise rotation

One polygon that has a particularly important role to play in this chapter is the **unit square** (see Figure 1.11). This is the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$; we shall usually denote it by the symbol S.

Activity 1.7 Rotating the unit square

Determine the matrix that represents the rotation $r_{\pi/4}$. Use it to find the image of the unit square S under $r_{\pi/4}$. Sketch the image.

A solution is given on page 56.

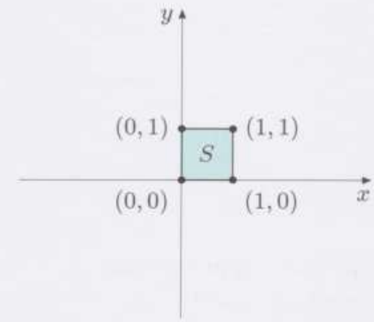


Figure 1.11 The unit square

Reflections

Reflections can be represented by matrices in much the same way as rotations. In Chapter A3, we discussed the reflection in the line through the origin that makes an angle θ with the positive x-axis (see Figure 1.12). The function that represents this reflection is

$$\begin{aligned} q_\theta\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x\cos(2\theta)+y\sin(2\theta), x\sin(2\theta)-y\cos(2\theta)). \end{aligned}$$

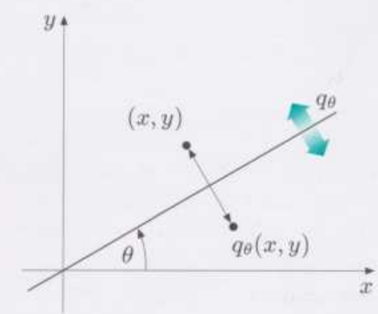


Figure 1.12 The reflection q_θ

In vector form, the image of $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ under the reflection q_θ is

$$\begin{pmatrix} x\cos(2\theta)+y\sin(2\theta) \\ x\sin(2\theta)-y\cos(2\theta) \end{pmatrix},$$

which can be written as

$$\begin{pmatrix} x\cos(2\theta)+y\sin(2\theta) \\ x\sin(2\theta)-y\cos(2\theta) \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{Q}_\theta \mathbf{x},$$

where

$$\mathbf{Q}_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

Matrix description of reflection

A **reflection** of the plane in a line through the origin that makes an angle θ with the positive x-axis has the form

$$\begin{aligned} q_\theta\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{Q}_\theta \mathbf{x}, \end{aligned}$$

where $\mathbf{Q}_\theta$ is the matrix $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$.

Activity 1.8 Reflecting the unit square

Determine the matrix that represents the reflection $q_{\pi/4}$. Use it to find the image of the unit square S under $q_{\pi/4}$.

A solution is given on page 57.

Although rotations and reflections are very different transformations of $\mathbb{R}^2$, they have the same form when expressed in terms of matrices, namely

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{A}\mathbf{x}. \end{aligned} \tag{1.2}$$

In the case of a rotation, $\mathbf{A} = \mathbf{R}_\theta$ for some θ, and in the case of a reflection, $\mathbf{A} = \mathbf{Q}_\theta$ for some θ.

In the next section, we explore some other functions that have the matrix form described in (1.2).

Summary of Section 1

This section has reviewed or introduced:

- the concept of a vector, and the use of arrows and components to represent vectors;
- the notation $\overrightarrow{PQ}$;
- the manipulation of vector equalities;
- the operations of vector addition, subtraction, taking the negative, and scalar multiplication;
- the use of vectors, in simple cases, to determine the position of a point on a figure;
- the concept of a position vector and its use to find the position of a point some given fraction of the way along a line segment;
- the vector description of translations, and the matrix descriptions of rotations and reflections, and their use to find images of points and polygons.

Exercises for Section 1

Exercise 1.1

Figure 1.13 shows a triangle OPQ that has one vertex at the origin O. The point X lies half-way along side OP, and the point Y lies a third of the way from X to Q. Find an expression for the position vector $\mathbf{y}$ of Y in terms of the position vectors $\mathbf{p}$ and $\mathbf{q}$ of P and Q.

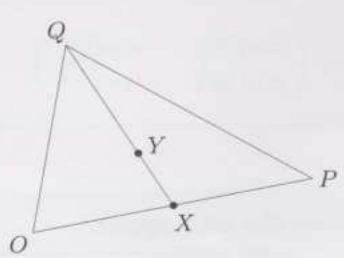


Figure 1.13 Locating X and Y

Exercise 1.2

Determine the matrix that represents $q_{\pi/6}$, the reflection of the plane in a line that passes through the origin and makes an angle $\pi/6$ measured anticlockwise from the positive x-axis. Use the matrix to find the image under this reflection of the triangle with vertices at $(1, \sqrt{3})$, $(\sqrt{3}, -1)$, $(-\sqrt{3}, 1)$.

Exercise 1.3

Identify each of the following matrices as a rotation matrix $\mathbf{R}_\theta$ or a reflection matrix $\mathbf{Q}_\theta$, and determine the angle θ in each case.

(a) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

2 Linear transformations

2.1 What is a linear transformation?

In the previous section, you saw that all rotations about the origin and all reflections in lines through the origin can be expressed as functions with rules of the form $\mathbf{x} \longmapsto \mathbf{Ax}$, where $\mathbf{A}$ is a 2×2 matrix. Any function with such a rule is called a *linear transformation*.

Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, for any positive integer n, may also be defined. In this course, we consider only the case $n = 2$.

> A **linear transformation** of the plane is a function of the form
>
> $$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{Ax}, \end{aligned}$$
>
> where $\mathbf{A}$ is a 2×2 matrix.
>
> The transformation f is said to be *represented* by the matrix $\mathbf{A}$.

For example, the function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x - 4y, 2x + 5y) \end{aligned} \tag{2.1}$$

is a linear transformation represented by the matrix

$$\begin{pmatrix} 3 & -4 \\ 2 & 5 \end{pmatrix},$$

because, in terms of vectors, the rule is

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 3x - 4y \\ 2x + 5y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since every linear transformation we deal with has domain and codomain $\mathbb{R}^2$, a linear transformation represented by a matrix $\mathbf{A}$ is often specified just by the phrase: the linear transformation $f(\mathbf{x}) = \mathbf{Ax}$.

Linear transformations preserve 'linearity'; that is, they map, or send, lines to lines. The following result, illustrated in Figure 2.1, makes this precise.

The linear transformation represented by the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ maps all points to the origin.

> **Linear transformations preserve linearity**
>
> Let f be a linear transformation represented by a 2×2 matrix $\mathbf{A}$, other than the zero matrix. Let ℓ be a line through a point P in the direction of a vector $\mathbf{u}$. Then the image $f(\ell)$ is a line through the point $f(P)$ in the direction of the vector $\mathbf{Au}$.

To see why this result holds, let Q be any point on ℓ (see Figure 2.1). Then the position vector of Q has the form

$$\mathbf{q} = \mathbf{p} + t\mathbf{u}, \quad \text{for some } t \in \mathbb{R}.$$

Thus the point $f(Q)$ has position vector

Here we are using properties of matrices given in the Study guide.

$$\mathbf{Aq} = \mathbf{A}(\mathbf{p} + t\mathbf{u}) = \mathbf{Ap} + t(\mathbf{Au}).$$

Since $\mathbf{Ap}$ is the position vector of the point $f(P)$, we deduce that $f(Q)$ lies on a line ℓ' through $f(P)$ in the direction of the vector $\mathbf{Au}$. As Q moves along ℓ, the image point $f(Q)$ moves along the line ℓ', so $\ell' = f(\ell)$.

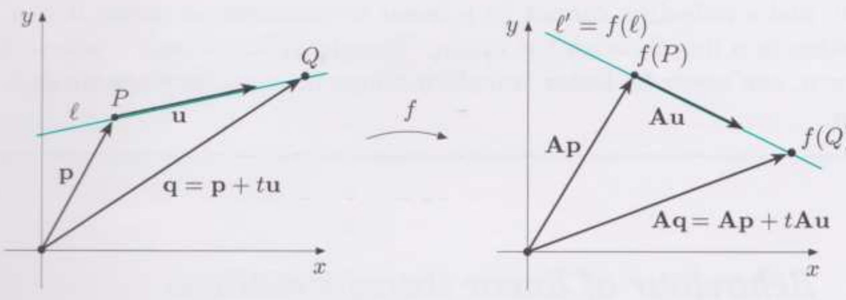


Figure 2.1 ℓ' is the image of ℓ

The above result also shows that any line parallel to the vector $\mathbf{u}$ is mapped by f to a line parallel to $\mathbf{Au}$; that is, linear transformations preserve 'parallelism'.

Linear transformations preserve parallelism

Let f be a linear transformation represented by a 2×2 matrix $\mathbf{A}$, other than the zero matrix. Let ℓ_1 and ℓ_2 be two parallel lines. Then the image lines $f(\ell_1)$ and $f(\ell_2)$ are also parallel.

Another important property of linear transformations is that they leave the origin fixed; for if $f(\mathbf{x}) = \mathbf{Ax}$ is any linear transformation, then

$$f(\mathbf{0}) = \mathbf{A0} = \mathbf{0}.$$

Here $\mathbf{0}$ is the position vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the origin.

This observation can often be used to recognise when a transformation of the plane is *not* linear. For example, the translation $t_{3,2}$ is not linear because it sends the origin to the point $(3, 2)$.

Activity 2.1 Checking for a linear transformation

For each of the following functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, either explain why f is not linear, or write down the matrix that represents f.

(a) $f(x, y) = (x + 2y, y - x)$.

(b) $f(x, y) = (x + y + 2, y - 2x)$.

(c) $f(x, y) = (0, 0)$.

(d) $f(x, y) = (2, -1)$.

(e) f reflects the plane in the line $x = 2$.

(f) f rotates the plane clockwise through $\frac{1}{4}\pi$ about the origin.

Solutions are given on page 57.

Comment

Notice that although isometries of the plane preserve linearity they cannot be linear transformations unless they fix the origin. In particular, a rotation cannot be a linear transformation unless it is a rotation about the origin, and a reflection cannot be a linear transformation unless it is a reflection in a line through the origin. Translations, through a non-zero vector $\mathbf{a}$, can never be linear transformations because they always shift the origin.

2.2 Behaviour of linear transformations

Having seen that linear transformations fix the origin and preserve linearity, we are now in a position to investigate the behaviour of linear transformations in more detail. A useful starting point is to consider what happens to the unit square under a linear transformation. Since linear transformations preserve parallelism, the image should be a parallelogram.

Activity 2.2 Image of the unit square

Let g be the linear transformation represented by the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, and let S be the unit square, as shown on the left of Figure 2.2.

(a) Find the images under g of the vertices of S. Hence sketch the image of S on the right-hand set of axes in Figure 2.2. Is the image a parallelogram?

(b) Do you notice any relationships between the vertices of the image and the matrix $\mathbf{A}$?

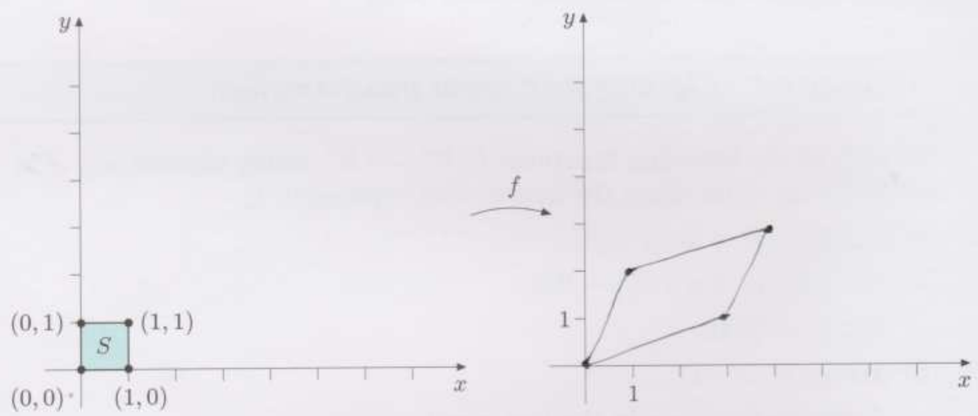


Figure 2.2 Finding the image of the square S

Solutions are given on page 57.

Activity 2.2 illustrates that it is straightforward to find the images of the points $(1,0)$ and $(0,1)$ under a linear transformation.

Images of $(1,0)$ and $(0,1)$

If f is the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

Thus f maps $(1,0)$ and $(0,1)$ to points whose position vectors are the first and second columns of $\mathbf{A}$, respectively.

But what about the images of other points? In the case of the linear transformation $g(\mathbf{x}) = \mathbf{A}\mathbf{x}$ in Activity 2.2, you saw that $(1,1)$ maps to a point whose position vector is the sum of the columns of $\mathbf{A}$. The following equations show how this idea generalises to give the image of any point (x,y) under a linear transformation f that is represented by a matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}. \tag{2.2}$$

So the image of any point (x,y) under f has position vector that we can obtain by adding x times the first column of $\mathbf{A}$ to y times the second column of $\mathbf{A}$.

Figure 2.3, overleaf, illustrates how this result can help us to visualise the behaviour of a linear transformation by showing its effect on the set G of all points with integer coordinates. For ease of reference, we shall refer to the corresponding grid of all lines parallel to the axes which pass through the points of G as the **unit grid**. The unit grid divides $\mathbb{R}^2$ into squares of sidelength 1 called **grid squares**.

We already know that f maps $(1,0)$ and $(0,1)$ to the points with position vectors $\mathbf{a} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b \\ d \end{pmatrix}$, respectively. So, since f preserves parallelism, the unit square S maps to the parallelogram S', shown in the figure.

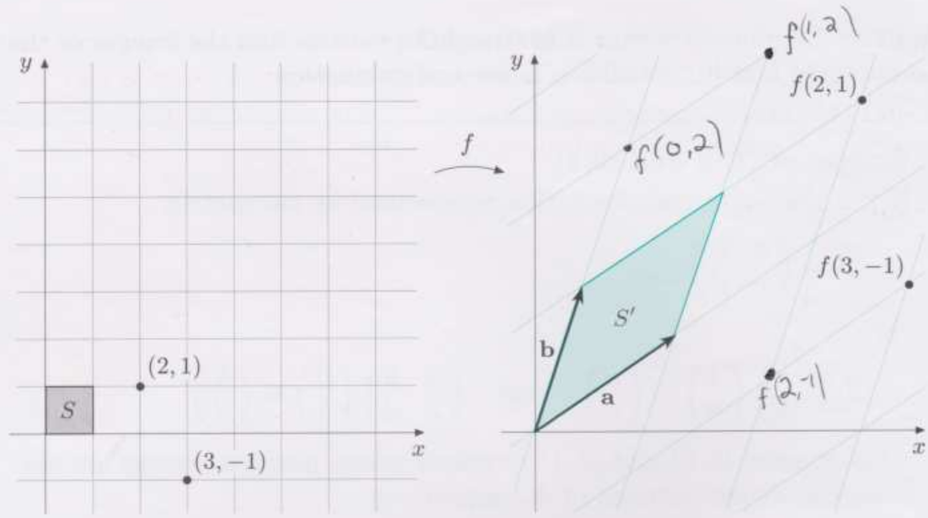


Figure 2.3 Image of S and G

In order to see what happens to G under f, green guide lines in the codomain that partition $\mathbb{R}^2$ into congruent parallelograms have been inserted. It turns out that the image of G is simply the set of points where these guide lines intersect. The reason for this is apparent from equation (2.2). For example, we know from equation (2.2) that $(2, 1)$ has image $f(2, 1)$ with position vector $2\mathbf{a} + \mathbf{b}$, so we can plot its position in the codomain by starting at the origin and counting 2 parallelogram 'lengths' in the direction of $\mathbf{a}$ followed by 1 parallelogram 'length' in the direction of $\mathbf{b}$. Similarly, we know that the image $f(3, -1)$ has position vector $3\mathbf{a} - \mathbf{b}$, so we can plot its position by starting at the origin and counting 3 parallelogram 'lengths' in the direction of $\mathbf{a}$ followed by 1 parallelogram 'length' in the direction opposite to $\mathbf{b}$.

Activity 2.3 Image of the unit grid

The coordinates (m, n) have been used for a grid point rather than (x, y) to emphasise that each such point has integer coordinates.

For each of the following grid points (m, n), use the above reasoning to plot the position of $f(m, n)$ on the codomain axes of Figure 2.3.

(a) $(1, 2)$ (b) $(2, -1)$ (c) $(0, 2)$

Solutions are given on page 58.

The effect of f on G, the set of grid points, in Figure 2.3 should now be clear. Each point (m, n) of G maps to a point that we locate by counting m parallelogram 'lengths' from the origin in the direction of $\mathbf{a}$, followed by n parallelogram lengths in the direction of $\mathbf{b}$. If m (or n) is negative, then the counting is performed in the direction opposite to $\mathbf{a}$ (or $\mathbf{b}$).

One of the most striking features illustrated by the image of the unit grid is the *homogeneous* nature of a linear transformation. By homogeneous we mean that the images of all the grid squares are congruent parallelograms that form a regular (though possibly skewed) image grid.

And there is no need to stop with the unit grid. For example, suppose that the unit grid is subdivided, like a sheet of graph paper, so that S and each of the other grid squares comprises 25 congruent, smaller squares. Then its image grid will be subdivided so that S' and each of the other grid parallelograms comprises 25 congruent, smaller parallelograms. Such a subdivision enables us to see what happens to points with non-integer coordinates (see Figure 2.4). Moreover, this process of subdivision can be repeated as often as necessary and the homogeneity property still holds.

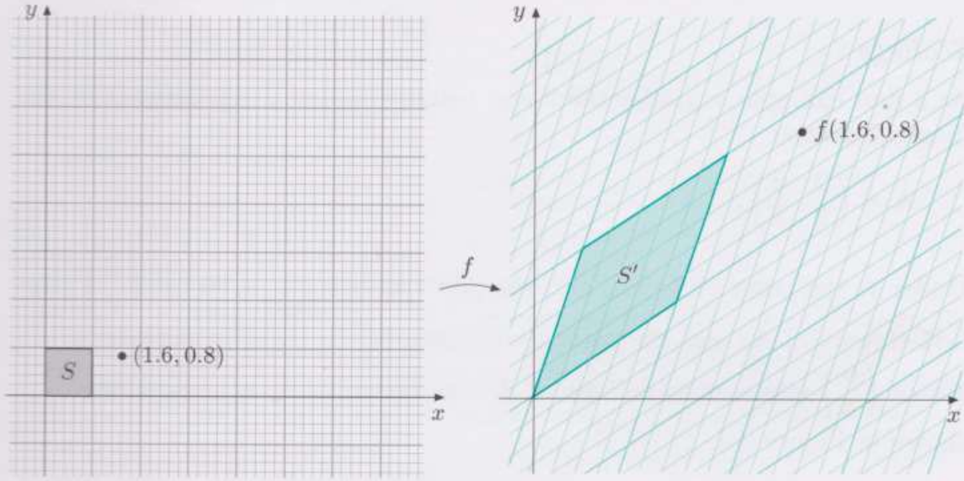


Figure 2.4 Refining the grid for points with non-integer coordinates

An immediate consequence of this homogeneity property is that a linear transformation f scales the area of all figures by the same factor (see Figure 2.5). For if F is a figure that surrounds N grid squares, then the image of F will be a figure that surrounds the N corresponding congruent parallelograms. The factor by which the area of F is scaled must therefore be equal to the area of S', for this is the image $f(S)$ of the square S that has unit area.

In Figure 2.5, $N = 11.5$.

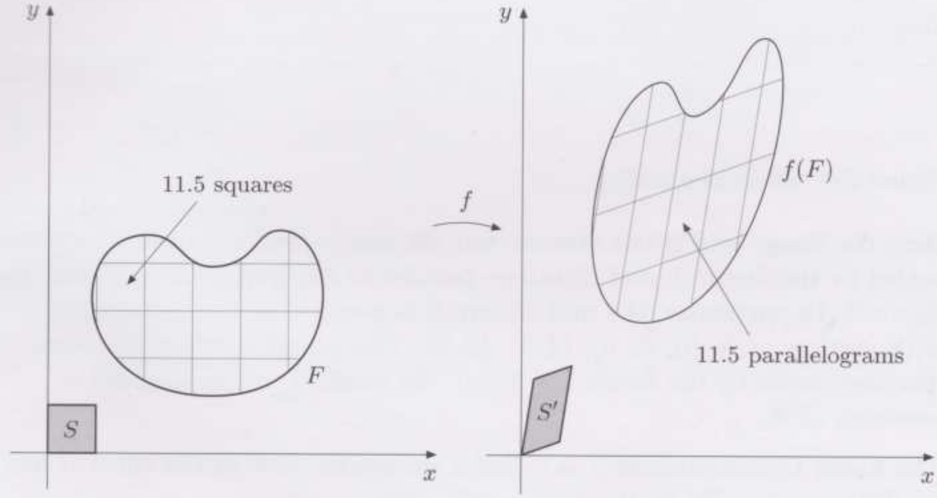


Figure 2.5 Scaling areas

This discussion is summarised in the following result.

Scaling areas of figures
Under a linear transformation f, the areas of figures are scaled by a factor equal to the area of the image $f(S)$ of the unit square S.

Having seen how a typical linear transformation behaves, let us now concentrate on some special cases.

Scalings

A square matrix is **diagonal** if all its non-zero elements lie on the *leading*, or *main*, diagonal (from top left to bottom right).

As a first example, consider the linear transformation f represented by the diagonal matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Under f, the point $(1,0)$ maps to the point with position vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (the first column of $\mathbf{A}$) and the point $(0,1)$ maps to the point with position vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ (the second column of $\mathbf{A}$), so the unit grid is mapped as shown in Figure 2.6.

The coloured arrows in Figure 2.6 indicate the scaling effects of f.

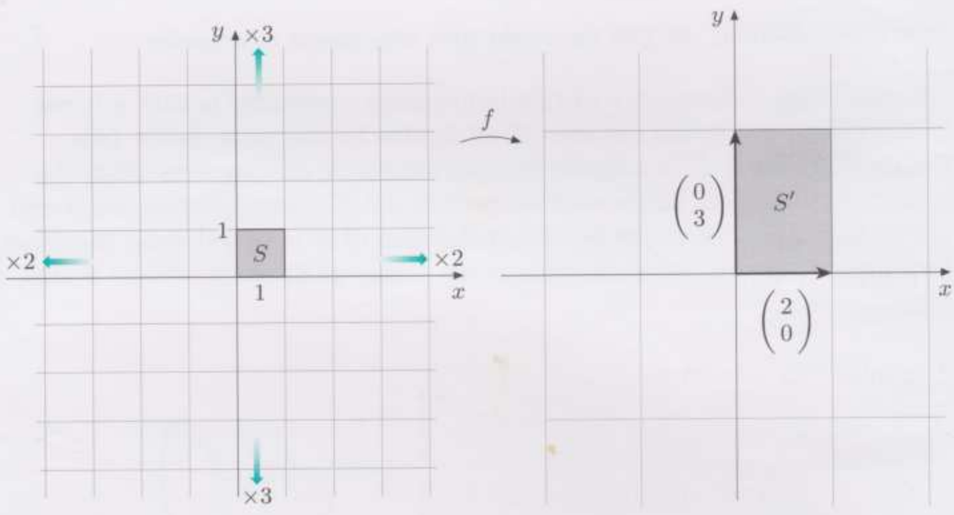


Figure 2.6 Effect of a scaling

Here the image grid is not skewed, but distances parallel to the x-axis are scaled by the factor 2, and distances parallel to the y-axis are scaled by the factor 3. In particular, the unit square S is mapped to the rectangle S' with vertices at $(0,0)$, $(2,0)$, $(2,3)$, $(0,3)$. The transformation therefore increases areas by the factor $2 \times 3 = 6$, the product of the diagonal elements of $\mathbf{A}$.

The linear transformation f is called a *scaling* because of the effect it has on distances parallel to the x-axis and y-axis.

What happens if the diagonal matrix representing a linear transformation has negative elements? You are asked to consider such cases in the next activity.

Activity 2.4 Some more diagonal matrices

For each of the matrices $\mathbf{A}$ below, sketch the image of the unit grid (as in Figure 2.6) under the linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. In each case, state whether the image grid is skewed and describe the effect of f on distances parallel to the x-axis and parallel to the y-axis.

Also, in each case, calculate the factor by which areas are changed and compare your answer with the product of the diagonal elements of $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$

Solutions are given on page 58.

Comment

In the solution to part (b), the location of S' in Figure S.14 indicates that part of the effect of f is to reflect in the y-axis. This effect is associated with the negative sign in the matrix. In Section 3, when an appropriate technique has been developed, this linear transformation, and that in part (c), will be revisited.

The above discussion leads to the following definition.

Matrix description of a scaling

The linear transformation

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{A}\mathbf{x}, \end{aligned}$$

where $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a \neq 0$, $b \neq 0$, is called a **scaling with factors** a and b.

The following properties of scalings were illustrated in Activity 2.4.

In general, if f is a scaling with factors a and b, then the image of the unit grid is not skewed. Distances parallel to the x-axis are scaled by the factor $|a|$, and distances parallel to the y-axis are scaled by the factor $|b|$.

The overall effect of a scaling with factors a and b is to cause areas to be scaled by the factor $|ab|$. If the product ab is negative, then the transformation involves a reflection (see the comment above). This changes the orientation of figures in the sense that any anticlockwise configuration of points maps to a clockwise configuration of points. For example, consider the scaling f in Activity 2.4(b). It follows from that activity that if the vertices of the unit square S are labelled anticlockwise as O, A, B, C, then their images under f are O, A', B', C', which is a clockwise configuration. See Figure 2.7, overleaf.

In general, any polygonal figure can be given an **orientation** – anticlockwise or clockwise – by specifying three or more points of the polygon in the appropriate order. Later in this section a criterion is given which distinguishes linear transformations that preserve orientation from those that reverse it.

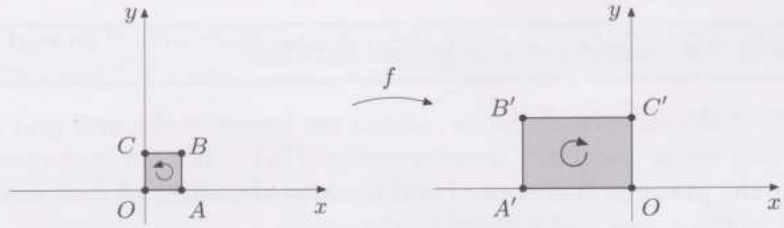


Figure 2.7 A change of orientation

Uniform scalings

One case that deserves a special mention is where the diagonal elements of the matrix representing a scaling are equal. We then have

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. In such a case, the linear transformation scales distances in both the x- and y-directions by the same factor, namely a. We say that the transformation is a **uniform scaling with factor** a. Figure 2.8 illustrates the uniform scaling with factor 2 given by the function

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto 2\mathbf{I}\mathbf{x}. \end{aligned}$$

Here, each point $\mathbf{x}$ moves outwards to the point $2\mathbf{x}$, thereby doubling distances, irrespective of their direction. Areas are scaled by the factor $2^2 = 4$.

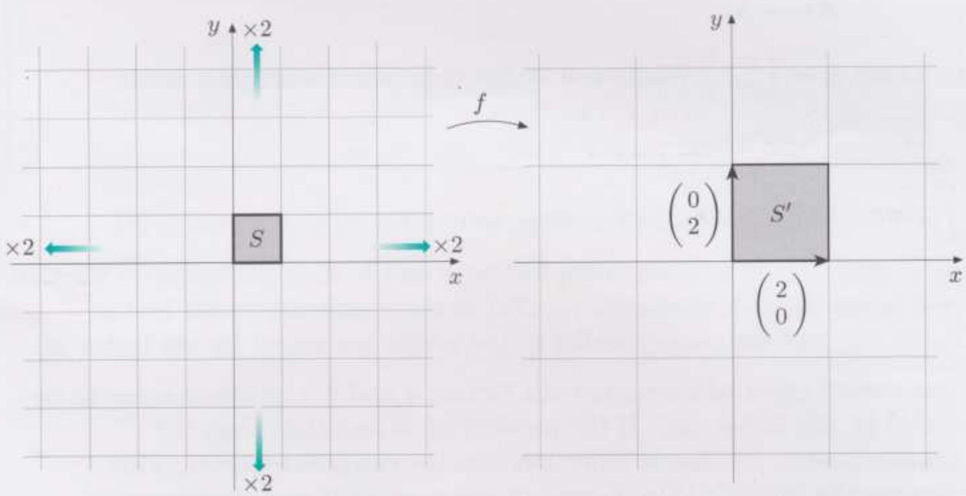


Figure 2.8 Effect of the uniform scaling with factor 2

In general, the linear transformation represented by the matrix $a\mathbf{I}$ scales all distances by the factor $|a|$ and scales areas by the factor a^2.

Shears

You have seen that scalings leave the unit grid unskewed. The next activity invites you to investigate a linear transformation, known as a *shear*, that *does* skew the grid.

This activity uses the formula for the area of a parallelogram, namely

$$area = ah,$$

where a is the length of the base and h is the height.

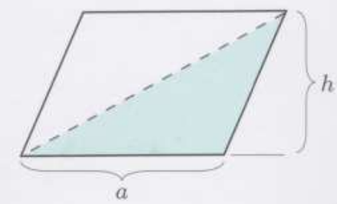


Figure 2.9 Area $= ah$

This formula is obtained by cutting the parallelogram along a diagonal into two congruent triangles, each of area $\frac{1}{2}ah$.

Activity 2.5 Interpreting a shear

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(a) Describe the geometric effect of f on the unit grid.

(b) Calculate the factor by which f changes areas. What effect does f have on the orientation of figures?

Solutions are given on page 59.

Comment

This activity illustrates an important property of shears, namely that they leave areas and orientation unchanged.

You have seen that the transformation in Activity 2.5 shears the plane by moving each point parallel to the x-axis through a distance that is proportional to the point's height above the axis. (Here height and distance are 'signed' so that points below the axis move in the opposite direction to those above it.) However this is not the only type of shear. Shears can be performed in any specified direction. Figure 2.10 illustrates the effect on some points of a shear parallel to a line ℓ.

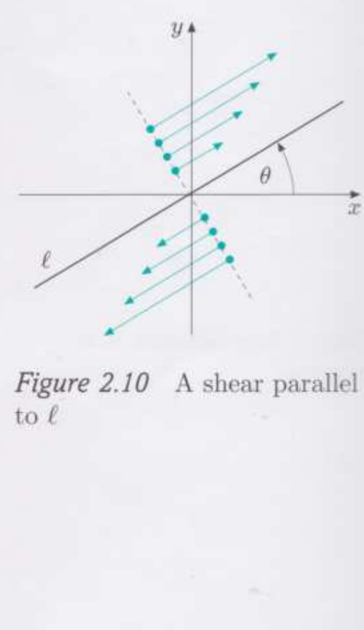


Figure 2.10 A shear parallel to ℓ

> A **shear parallel to a line** ℓ is a linear transformation of the plane that shifts each point P parallel to ℓ, through a distance that is proportional to the perpendicular distance of P from ℓ. Points on opposite sides of ℓ shift in opposite directions, whereas points on the same side of ℓ shift in the same direction.

In particular, a shear parallel to the x-axis is represented by a matrix of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in \mathbb{R}$, and is called an ***x*-shear with factor *a***. (In Activity 2.5, $a = 2$.) Similarly, a shear parallel to the y-axis has matrix of the form $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, $a \in \mathbb{R}$, and is called a ***y*-shear with factor *a***. In both cases, points that lie at a distance d from the axis are shifted through a distance ad parallel to the axis. The x-shear and the y-shear are the only types of shear we shall consider.

2.3 Determinants, areas and orientation

What effect does a general linear transformation have on areas, and how does it affect orientation? Figure 2.11 illustrates the behaviour of a linear transformation f represented by a matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To keep track of orientation, the vertices of the unit square S have been labelled with the letters O, P, Q, R in an anticlockwise direction, and their images have been labelled O, P', Q', R', respectively. If, as shown in the figure, the order of these images is anticlockwise, then f preserves orientation; if, however, the order of the images turns out to be clockwise, then f reverses orientation.

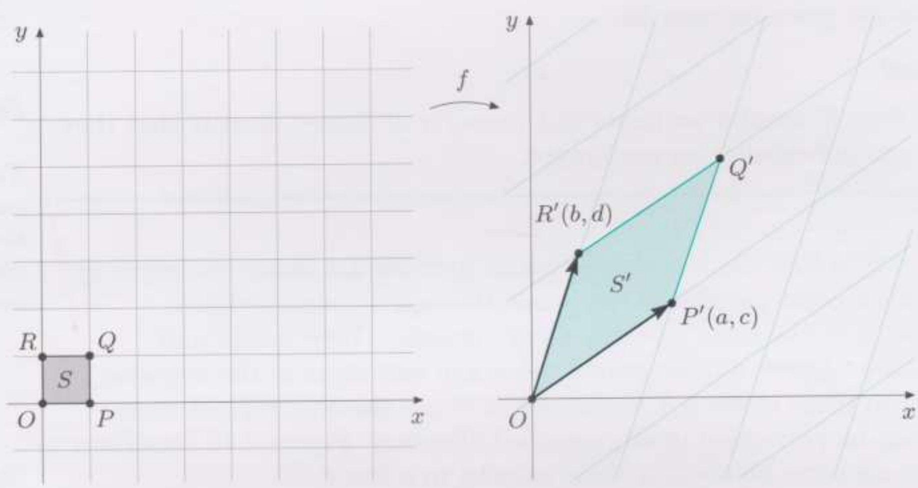


Figure 2.11 Labelling vertices

To calculate the factor by which a general linear transformation scales areas, we need to express the area of parallelogram $OP'Q'R'$ in terms of the matrix elements a, b, c, d. One approach, illustrated in Figure 2.12, is to rotate $OP'Q'R'$ about the origin to give a congruent parallelogram $OP''Q''R''$ that has P'' on the positive x-axis. Then $OP'' = OP' = l$, say, and the area of $OP''Q''R''$ (and hence of $OP'Q'R'$) is

Since l is a distance, it is positive.

$$l \times h,$$

where h is the modulus of the y-coordinate of R''.

To complete the calculation we need to locate R''. Suppose that θ is the angle measured from the positive x-axis to OP'. Then the rotation that sends $OP'Q'R'$ to $OP''Q''R''$ is $r_{-\theta}$ (see Figure 2.12).

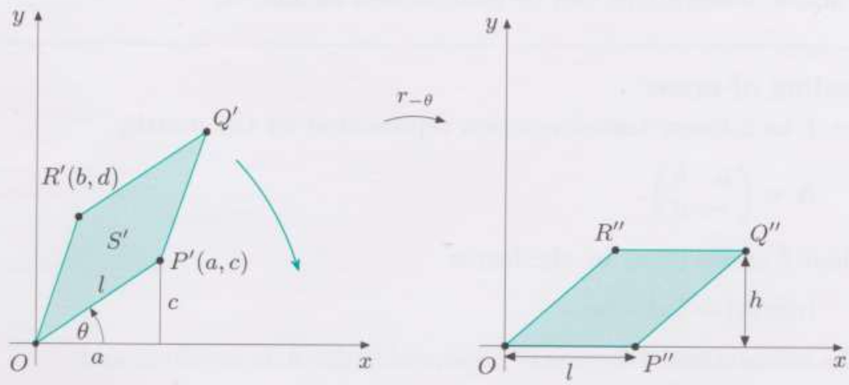


Figure 2.12 Rotating S'

From Figure 2.12, we have

$$\cos(-\theta) = \cos\theta = a/l \quad \text{and} \quad \sin(-\theta) = -\sin\theta = -c/l,$$

so $r_{-\theta}$ is represented by the matrix

$$\mathbf{R}_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} a/l & c/l \\ -c/l & a/l \end{pmatrix}.$$

Since $\begin{pmatrix} b \\ d \end{pmatrix}$ is the position vector of R', it follows that $R'' = r_{-\theta}(R')$ has position vector

$$\begin{pmatrix} a/l & c/l \\ -c/l & a/l \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} (ab+cd)/l \\ (-cb+da)/l \end{pmatrix} = \begin{pmatrix} (ab+cd)/l \\ (ad-bc)/l \end{pmatrix}.$$

So $h = |ad - bc|/l$. The area of $OP'Q'R'$ is therefore

$$\begin{aligned} l \times h &= l \times |ad - bc|/l \\ &= |ad - bc| = |\det \mathbf{A}|. \end{aligned}$$

So, since S has area 1, areas are scaled under f by the modulus of the determinant of $\mathbf{A}$, $|\det \mathbf{A}|$.

The determinant of a 2×2 matrix was defined in MST121 Chapter B2, Subsection 5.2

But what about the sign of $\det \mathbf{A}$? Since l is positive, the determinant has the same sign as the y-coordinate of R'', namely $(ad - bc)/l$. Thus if $\det \mathbf{A}$ is positive, then R'' lies above the x-axis and $OP''Q''R''$ has the same (anticlockwise) orientation as the unit square $OPQR$ (see Figure 2.13(b)). On the other hand, if $\det \mathbf{A}$ is negative, then R'' lies below the x-axis, and $OP''Q''R''$ has opposite (clockwise) orientation to $OPQR$ (see Figure 2.13(c)).

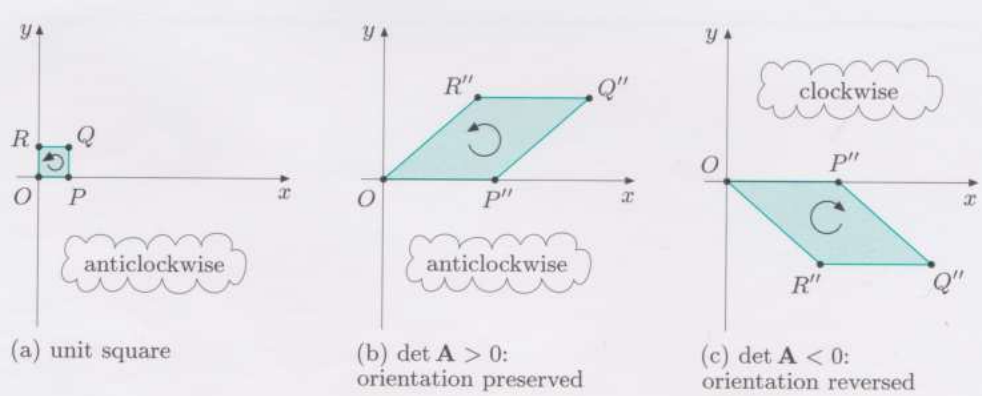


Figure 2.13 Interpreting the sign of det **A**

The above observations can be summarised as follows.

Scaling of areas

Let f be a linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then f scales areas by the factor

$$|\det \mathbf{A}| = |ad - bc|.$$

The orientation of figures is preserved if $\det \mathbf{A}$ is positive, and reversed if $\det \mathbf{A}$ is negative.

The case where $\det \mathbf{A} = 0$ is discussed on page 31.

The following activity illustrates the use of determinants to calculate the area of a triangle that has one vertex at the origin. You will see more general examples in Sections 3 and 4.

Activity 2.6 Area of a triangle

Let f be the linear transformation that sends $(1,0)$ to $(2,5)$ and $(0,1)$ to $(3,1)$. Write down the matrix that represents f and calculate its determinant. Hence calculate the area of the triangle T with vertices at $(0,0)$, $(2,5)$, $(3,1)$.

A solution is given on page 60.

Since isometries that fix the origin are linear transformations that preserve both the size and shape of figures, they should certainly preserve the area of figures. The following activity asks you to verify this in two special cases.

Activity 2.7 Determinants of rotation and reflection matrices

(a) Write down the general form of the matrix that represents a rotation r_θ, and calculate its determinant. Is the value of the determinant consistent with how you expect the rotation to behave?

(b) Write down the general form of the matrix that represents a reflection q_θ, and calculate its determinant. Is the value of the determinant consistent with how you would expect the reflection to behave?

Solutions are given on page 60.

Flattening the plane

Next we consider some linear transformations with rule $\mathbf{x} \longmapsto \mathbf{Ax}$ for which $\det \mathbf{A} = 0$. A zero determinant suggests that all areas are reduced to zero under the transformation, and this in turn suggests that the transformation collapses the plane in some way.

A simple case is the so-called **zero transformation** represented by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Under the zero transformation all points map to $(0, 0)$, so the entire plane collapses onto a single point, as indicated in Figure 2.14. (Notice that in figures like these it seems sensible to combine the domain and codomain and illustrate the collapse on a single diagram.)

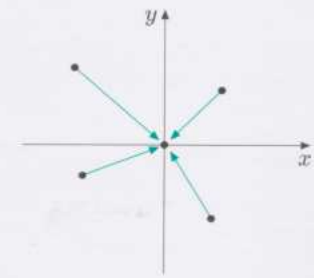


Figure 2.14 Collapsing onto $(0, 0)$

There are other ways in which linear transformations can collapse the plane. For example, consider the linear transformation f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}.$$

Since the position vectors of the images of $(1, 0)$ and $(0, 1)$ are

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

they lie in the same direction. So the unit grid (and hence the entire domain $\mathbb{R}^2$) collapses onto the line that passes through the points $(0, 0)$, $(4, 2)$ and $(6, 3)$. This line has equation $y = \frac{1}{2}x$, or equivalently, $x - 2y = 0$ (see Figure 2.15).

In fact, the image of any point (x, y) has the position vector given by

$$\begin{aligned} \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 4 \\ 2 \end{pmatrix} + y\begin{pmatrix} 6 \\ 3 \end{pmatrix} &= 2x\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3y\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= (2x + 3y)\begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.3}$$

So all points of the domain $\mathbb{R}^2$ that lie on the line $2x + 3y = 0$ collapse onto the point $(0, 0)$; all points on the line $2x + 3y = 5$ collapse onto the point with position vector $5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$; and in general, all points on the line $2x + 3y = k$ collapse onto the point with position vector $k\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So, overall, the plane is flattened onto the line $x - 2y = 0$.

In MST121, Chapter A2, the equation of the line $2x + 3y = 0$ would have been written in the form $y = -\frac{2}{3}x$, the equation of the line $2x + 3y = 5$ would have been written $y = -\frac{2}{3}x + \frac{5}{3}$, and so on.

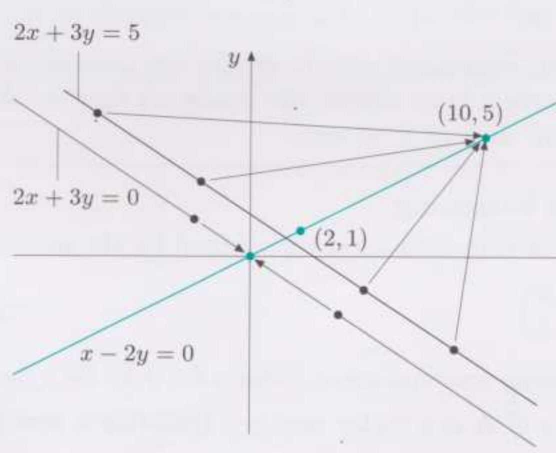


Figure 2.15 A flattening of the plane onto a line

Activity 2.8 A flattening of the plane

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}.$$

(a) Show that the images $f(1,0)$ and $f(0,1)$ lie on the same line through the origin.

(b) Express the position vector of the image of an arbitrary point (x, y) under f as a multiple of a single vector (as in equation (2.3)).

(c) Draw a diagram (similar to Figure 2.15) that shows the effect of f on the plane.

(d) Calculate the determinant of $\mathbf{A}$.

Solutions are given on page 60.

The linear transformations represented by the matrices $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}$ are examples of *flattenings*.

A linear transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that maps $\mathbb{R}^2$ onto a line through the origin or onto the origin is called a **flattening**.

In general, we can check whether the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a flattening by examining the image $S' = f(S)$ of the unit square S (see Figure 2.16). The condition for f to be a flattening is that f collapses the plane onto a single line (or in the case of the zero transformation onto the origin). For this to happen, one of the two position vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ along the edges of S' must be a scalar multiple of the other. But these two position vectors are the columns of $\mathbf{A}$, so the condition for f to be a flattening is that one column of $\mathbf{A}$ is a scalar multiple of the other.

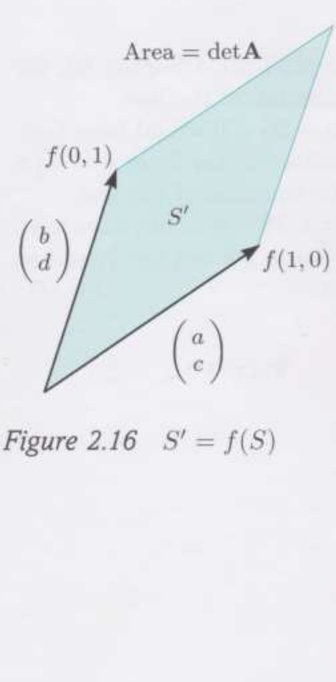


Figure 2.16 $S' = f(S)$

An alternative, but equivalent, way to ensure the collapse of S onto a single line or the origin is to impose the condition that $\det \mathbf{A} = 0$, for this will ensure that the area of S' is zero.

Criteria for a flattening

Let f be a linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the following are equivalent criteria for f to be a flattening:

◇ one column of $\mathbf{A}$ is a scalar multiple (possibly a zero multiple) of the other;

◇ the determinant of $\mathbf{A}$ is zero.

Earlier you saw that if f is a linear transformation represented by a matrix $\mathbf{A}$, then the effect that f has on orientation is determined by the sign of $\det \mathbf{A}$. A positive determinant indicates that f preserves orientation, whereas a negative determinant indicates that f reverses orientation. Now it is possible to see what happens when $\det \mathbf{A} = 0$. In such cases, f collapses the unit square onto a line (or the origin), so it no longer makes sense to talk about the orientation of $S' = f(S)$. A zero determinant indicates that f destroys orientation.

In Section 1, you saw the matrix representations of the rotation r_θ and the reflection q_θ. You now have all the tools needed to recognise scalings, x- and y-shears, and flattenings of the plane.

Activity 2.9 Recognising linear transformations

Identify the type of the linear transformation f represented by each of the matrices $\mathbf{A}$ given below. In each case, describe briefly the geometric effect of f. Also, in each case, calculate the factor by which areas are scaled and state whether orientation is preserved.

(a) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$

Solutions are given on page 60.

One-one and onto functions

A glance at Figures 2.14 and 2.15 reveals some important differences between flattenings and other linear transformations. A flattening sends many different points to a single image point. For example, the transformation illustrated in Figure 2.15 sends all the points on the line $2x + 3y = 0$ to the origin, all the points on the line $2x + 3y = 5$ to $(10, 5)$, and so on. Such functions are said to be *many-one*. By contrast, linear transformations such as isometries and shears are *one-one* functions in the sense that distinct points map to distinct points.

The following definitions are applicable to any type of function.

> A function $f\colon A \longrightarrow B$ is **one-one** or **one-to-one** if each element of $f(A)$ is the image of exactly one element of A; that is,
>
> for all $a, b \in A$, if $a \neq b$, then $f(a) \neq f(b)$.
>
> A function that is not one-one is **many-one** or **many-to-one**.

If f is a linear transformation, then $A = B = \mathbb{R}^2$ and a and b are points of $\mathbb{R}^2$.

To show that a function is many-one it is sufficient to exhibit just two points in the domain that map to the same point in the codomain. Thus we can show that the linear transformation f illustrated in Figure 2.15 is many-one by observing that $f(3, -2) = f(0, 0)$.

$f(\mathbf{x}) = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \mathbf{x}$

The following example illustrates one way to prove that a linear transformation is one-one (when it is). This method of proof is based on the following characterisation of a one-one function:

for all $a, b \in A$, if $f(a) = f(b)$, then $a = b$,

which is equivalent to that given in the definition box above.

Example 2.1 Checking whether a function is one-one

Let f be the linear transformation represented by the matrix

$$\begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}.$$

Show that f is one-one.

Solution

Suppose (r, s) and (u, v) are points such that $f(r, s) = f(u, v)$. Then

$$\begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}; \quad \text{that is,} \quad \begin{pmatrix} r + 6s \\ 3s \end{pmatrix} = \begin{pmatrix} u + 6v \\ 3v \end{pmatrix}.$$

Equating components, we obtain $r + 6s = u + 6v$ and $3s = 3v$, from which we conclude that $s = v$, and hence $r = u$. It follows that $(r, s) = (u, v)$. Hence f is one-one.

A second difference between flattenings and other linear transformations is that flattenings have image sets that fail to occupy the entire plane. For example, the image set of the flattening illustrated in Figure 2.15 is the line $x - 2y = 0$. By contrast, linear transformations such as isometries and shears have image sets that coincide with the codomain $\mathbb{R}^2$.

When the image set of a function f coincides with the codomain of f, we say that f is an *onto* function. We make the following definition.

> A function $f: A \longrightarrow B$ is **onto** if $f(A) = B$.

To show that a function $f: A \longrightarrow B$ is onto, we have to show that for each b in the codomain B there is a point a in A such that $f(a) = b$.

Example 2.2 Checking whether a function is onto

Let f be the linear transformation represented by the matrix

$$\begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}.$$

Show that f is onto.

Solution

Let (u, v) be an arbitrary point in the codomain $\mathbb{R}^2$. For (u, v) to be the image of a point (x, y) in the domain, we require $f(x, y) = (u, v)$; that is,

$$\begin{pmatrix} 1 & 6 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{or, equivalently,} \quad \begin{pmatrix} x + 6y \\ 3y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Solving for (x, y), we obtain $y = \frac{1}{3}v$ and hence $x = u - 6y = u - 2v$. So one point that maps to (u, v) is $(x, y) = (u - 2v, \frac{1}{3}v)$. Since (u, v) is an arbitrary point in the codomain $\mathbb{R}^2$, we conclude that $f(\mathbb{R}^2) = \mathbb{R}^2$. Hence f is onto.

In the following activity, you are asked to show that a particular linear transformation is one-one and onto.

Activity 2.10 Checking whether a function is one-one and onto

Let f be the linear transformation represented by the matrix

$$\begin{pmatrix} 0 & 2 \\ 3 & 4 \end{pmatrix}.$$

Show that f is (a) one-one (b) onto.

A solution is given on page 61.

Summary of Section 2

This section has introduced:

- the definition of a linear transformation of the plane, and the matrix that represents it;
- the use of the unit grid and unit square to visualise the behaviour of a linear transformation;
- particular types of linear transformations including scalings, shears and flattenings;
- the relationship of the determinant of a matrix to area and orientation of figures.

Exercises for Section 2

Exercise 2.1

Identify the type of the linear transformation f represented by each of the matrices $\mathbf{A}$ given below. In each case, describe briefly the geometric effect of f. Also, in each case, calculate the factor by which areas are scaled and state whether orientation is preserved.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 8 & 6 \\ 4 & 3 \end{pmatrix}$

Exercise 2.2

Let f be the linear transformation that sends $(1, 0)$ to $(-1, 4)$ and $(0, 1)$ to $(2, 3)$. Write down the matrix that represents f and use it to calculate the area of the triangle T with vertices at $(0, 0)$, $(-1, 4)$, $(2, 3)$.

Exercise 2.3

For each of the following linear transformations f, write down the matrix $\mathbf{A}$ that represents the transformation.

(a) f scales the plane by a factor of 3 in the direction of the x-axis, and by a factor 7 in the direction of the y-axis.

(b) f rotates the plane about the origin through $\pi/6$ in a clockwise direction.

(c) f shears the plane parallel to the x-axis in such a way that points at height 1 above the x-axis shift 4 units to the left.

(d) f maps the points $(1, 0)$ and $(0, 1)$ to the points $(2, 3)$ and $(-5, 4)$, respectively.

3 Composite and inverse transformations

3.1 Composite transformations

In Chapter A3, you saw that if one isometry f is followed by another isometry g, then the overall effect is a *composite* isometry $g \circ f$. For example, if the rotation $r_{\pi/3}$ is followed by the rotation $r_{\pi/6}$, then the overall effect is the rotation $r_{\pi/2}$; that is, $r_{\pi/2} = r_{\pi/6} \circ r_{\pi/3}$, as illustrated in Figure 3.1.

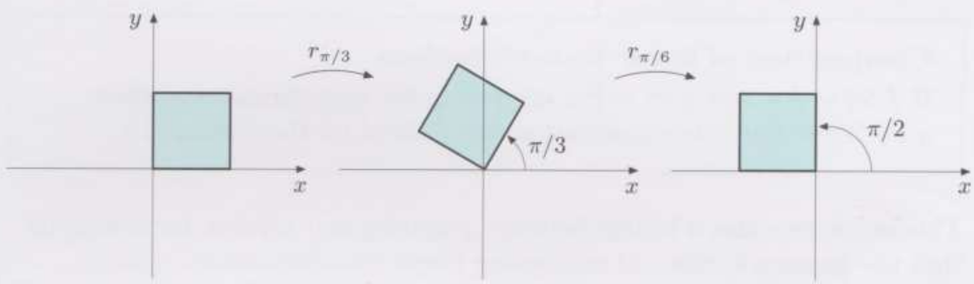
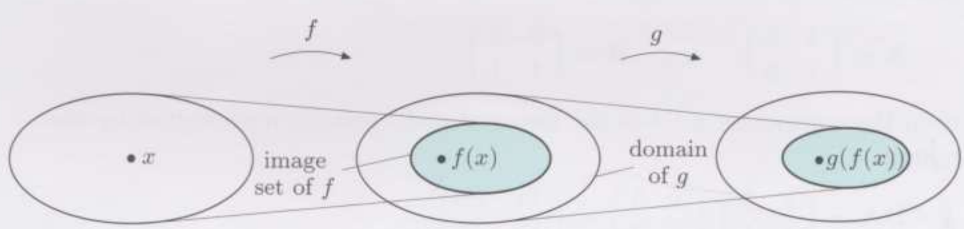


Figure 3.1 Composing rotations

Later, in Chapter B1, you saw that forming composites is not restricted to isometries. The composite $g \circ f$ can be formed from any two functions f and g provided that the image set of f is a subset of the domain of g.



Remember that $g \circ f$ means f followed by g.

Figure 3.2 Condition for a composite $g \circ f$ to exist

In this chapter, we concentrate on composites $g \circ f$, where f and g are linear transformations of the plane. Such composites can always be formed because the image set of f is a subset of $\mathbb{R}^2$, and $\mathbb{R}^2$ is the domain of g.

The following activity illustrates a correspondence between the *composite* of two linear transformations and matrix *multiplication*.

Activity 3.1 Matrix representing a composite of rotations

(a) Write down the matrices $\mathbf{R}_{\pi/3}$, $\mathbf{R}_{\pi/6}$, $\mathbf{R}_{\pi/2}$ that represent the linear transformations $r_{\pi/3}$, $r_{\pi/6}$, $r_{\pi/2}$, respectively.

(b) Verify that $\mathbf{R}_{\pi/2} = \mathbf{R}_{\pi/6}\mathbf{R}_{\pi/3}$.

(c) What can you deduce from the pair of equations

$$r_{\pi/2} = r_{\pi/6} \circ r_{\pi/3} \quad \text{and} \quad \mathbf{R}_{\pi/2} = \mathbf{R}_{\pi/6}\mathbf{R}_{\pi/3}?$$

Solutions are given on page 61.

The correspondence in Activity 3.1 is no accident, for if

$$f(\mathbf{x}) = \mathbf{Ax} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{Bx}$$

are two linear transformations of the plane (represented by matrices $\mathbf{A}$ and $\mathbf{B}$, respectively), and if $\mathbf{x}$ is the position vector of an arbitrary point in the plane, then

$$\begin{aligned}
(g \circ f)(\mathbf{x}) &= g(f(\mathbf{x})) && \text{(definition of composite)}\\
&= g(\mathbf{Ax}) && \text{(definition of } f)\\
&= \mathbf{B}(\mathbf{Ax}) && \text{(definition of } g)\\
&= (\mathbf{BA})\mathbf{x}. && \text{(a property of matrix multiplication)}
\end{aligned}$$

So the composite function $g \circ f$ is a linear transformation represented by the product matrix $\mathbf{BA}$.

Composition of linear transformations

If $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{x}) = \mathbf{Bx}$ are two linear transformations, then $g \circ f$ is the linear transformation represented by the matrix $\mathbf{BA}$.

This result provides a bridge between *geometry* and *algebra*, for it tells us that the geometric effect of composing linear transformations can be expressed algebraically in terms of matrix multiplication. From a theoretical point of view, this is important because it enables us to use matrix algebra to study linear transformations. More practically, it provides a systematic approach to the manipulation of linear transformations that is easily handled (particularly by a computer). For example, if $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{x}) = \mathbf{Bx}$ are linear transformations represented by the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix},$$

then the composite $g \circ f$ is the linear transformation represented by the matrix

$$\mathbf{BA} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 19 \\ 18 & 23 \end{pmatrix}.$$

Activity 3.2 Matrix of a composite

Let $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{x}) = \mathbf{Bx}$ be the x- and y-shears represented by the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively.

(a) Determine the matrices that represent each of the following composite transformations.

(i) $g \circ f$ (ii) $f \circ g$

(b) Which of the composites in part (a) is illustrated by Figure 3.3?

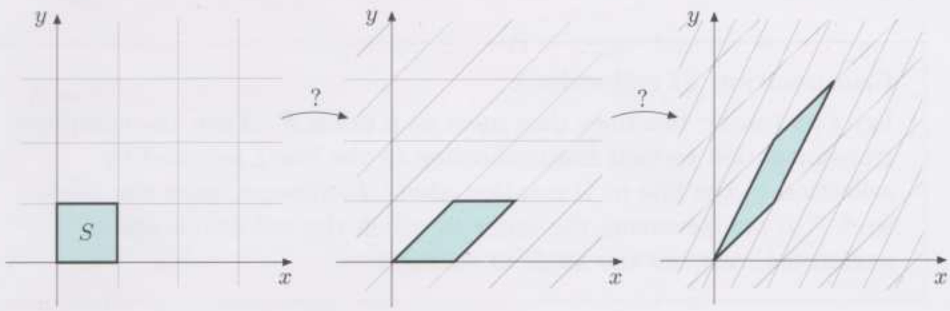


Figure 3.3 A composite transformation

Solutions are given on page 61.

The interplay between the algebra of matrices and the geometry of transformations provides us with two quite different ways to think about composites. For example, the result

$$r_\theta \circ r_\phi = r_{\theta+\phi},$$

is reasonably easy to visualise geometrically. But how about the equivalent result for reflections? Can you imagine what the composite $q_\theta \circ q_\phi$ looks like geometrically? Perhaps you can, but if not you can always use matrices to find $q_\theta \circ q_\phi$.

You have already seen that the reflections q_θ and q_ϕ are represented by the matrices

$$\mathbf{Q}_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad \text{and} \quad \mathbf{Q}_\phi = \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix},$$

respectively. So the composite $q_\theta \circ q_\phi$ is is represented by the product matrix

$$\begin{aligned} \mathbf{Q}_\theta\mathbf{Q}_\phi &= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\theta - 2\phi) & -\sin(2\theta - 2\phi) \\ \sin(2\theta - 2\phi) & \cos(2\theta - 2\phi) \end{pmatrix}. \end{aligned}$$

Here, we use the trigonometric difference formulas derived in Chapter A3, Subsection 3.2.

But this is the matrix that represents a rotation about the origin through the angle $2\theta - 2\phi = 2(\theta - \phi)$. So we conclude that $\mathbf{Q}_\theta\mathbf{Q}_\phi = \mathbf{R}_{2(\theta-\phi)}$ and hence that

$$q_\theta \circ q_\phi = r_{2(\theta-\phi)}. \tag{3.1}$$

In particular, with $\phi = 0$ and $\theta = \pi/2$, we see that a reflection in the x-axis followed by a reflection in the y-axis is equivalent to a rotation through π.

Here $\theta - \phi$ is the angle between the two lines of reflection and the composite is a rotation through twice this angle. In fact, this interpretation holds for reflections in *any* two lines that intersect. This is because we can always pick Cartesian axes that have the origin at the intersection point, and then apply the above matrix argument to derive the following general result, which is illustrated in Figure 3.4, overleaf.

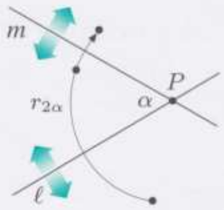


Figure 3.4 Composing reflections

Composition of reflections

Let ℓ and m be two lines that meet at a point P. Then the composite transformation formed from reflection in the line ℓ followed by reflection in the line m is rotation about P through twice the angle from ℓ to m. Reversing the order in which the reflections are performed, reverses the angle of rotation.

Next we consider the composite of a reflection and a rotation.

Activity 3.3 Composite of a reflection and a rotation

(a) Use matrix multiplication to show that the composite $r_\theta \circ q_\phi$ is a reflection in a certain line ℓ through the origin.

(b) What angle does ℓ make with the positive x-axis?

Solutions are given on page 62.

As a further illustration of the interplay between geometry and algebra, we consider what happens to orientation and areas under the composite $g \circ f$ of the linear transformations

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{B}\mathbf{x}.$$

There are two ways of thinking about the composite $g \circ f$ (see Figure 3.5). The first is to think of it as a single linear transformation represented by the matrix $\mathbf{BA}$. Under $g \circ f$, areas are scaled by the factor

$$|\det(\mathbf{BA})|.$$

The second way is to think of the transformation as a two-stage process. First f scales areas by the factor $|\det \mathbf{A}|$, and then g scales the result by a further factor $|\det \mathbf{B}|$. Overall, therefore, the composite transformation scales areas by the factor

$$|\det \mathbf{B}| \times |\det \mathbf{A}| = |\det \mathbf{B} \det \mathbf{A}|.$$

Thus

$$|\det(\mathbf{BA})| = |\det \mathbf{B} \det \mathbf{A}|. \tag{3.2}$$

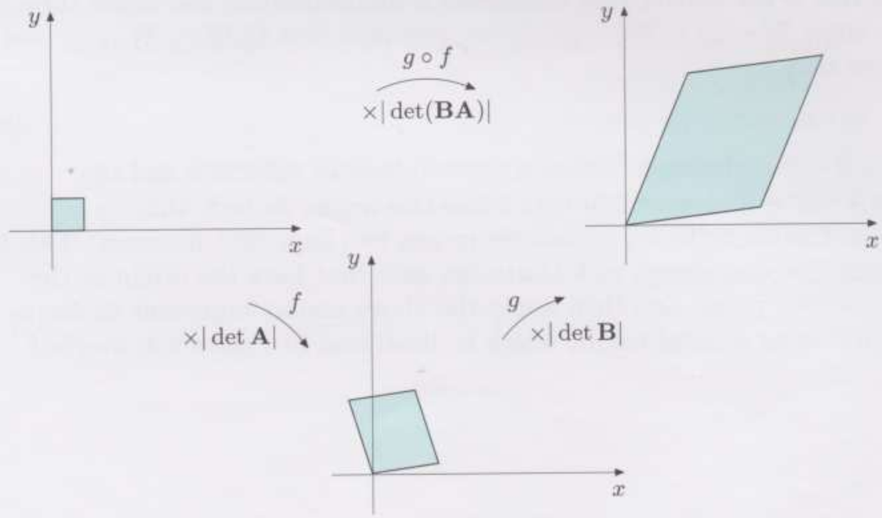


Figure 3.5 The effect of a composite linear transformation on areas

But what about the signs of $\det(\mathbf{BA})$ and $\det\mathbf{B}\det\mathbf{A}$? If f and g both preserve orientation, so will $g\circ f$. If f and g both reverse orientation, then $g\circ f$ will preserve orientation. Also, if g reverses the orientation reversed by f, then $g\circ f$ will preserve orientation. In terms of determinants, this means that

Recall that the orientation of figures is preserved if $\det\mathbf{A}$ is positive, and reversed if $\det\mathbf{A}$ is negative.

$$\det(\mathbf{BA}) \text{ is positive if } \begin{cases} \det\mathbf{A} \text{ and } \det\mathbf{B} \text{ are positive} \\ \det\mathbf{A} \text{ and } \det\mathbf{B} \text{ are negative.} \end{cases}$$

Otherwise, if f and g have opposite effects, then $g\circ f$ will reverse orientation. So

$$\det(\mathbf{BA}) \text{ is negative if } \det\mathbf{A} \text{ and } \det\mathbf{B} \text{ have opposite signs.}$$

Thus

$$\det(\mathbf{BA}) \text{ has the same sign as } \det\mathbf{B}\det\mathbf{A}. \tag{3.3}$$

On combining equations (3.2) and (3.3), we obtain the following result.

Determinant of a product matrix

Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices. Then

$$\det(\mathbf{BA}) = \det\mathbf{B}\det\mathbf{A}.$$

This states that the determinant of a product is the product of the determinants. Note that no such result holds for the determinant of the sum of two matrices.

As an illustration, recall from page 36 that if

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} \quad\text{and}\quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \quad\text{then}\quad \mathbf{BA} = \begin{pmatrix} 14 & 19 \\ 18 & 23 \end{pmatrix}.$$

These matrices have determinants

$$\det\mathbf{A} = 4\times 3 - 5\times 2 = 2,$$
$$\det\mathbf{B} = 2\times 1 - 3\times 4 = -10,$$
$$\det(\mathbf{BA}) = 14\times 23 - 19\times 18 = 322 - 342 = -20,$$

respectively. Notice, however, that we could calculate the third determinant more easily by using the rule for the determinant of a product:

$$\det(\mathbf{BA}) = \det\mathbf{B}\det\mathbf{A} = (-10)\times 2 = -20.$$

You will need to use the above rule in the following activity.

Activity 3.4 Composites of flattenings

Let f and g be linear transformations of the plane represented by matrices $\mathbf{A}$ and $\mathbf{B}$ respectively.

(a) Show that if g is a flattening, then the composite linear transformation $g\circ f$ is also a flattening.

(b) Show that if $g\circ f$ is a flattening, then either g or f (or both) is a flattening.

Solutions are given on page 62.

Scalings revisited

Here we use composition to complete the geometric interpretation of the two linear transformations considered in parts (b) and (c) of Activity 2.4. The scaling f in part (b) is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since

$$\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

the linear transformation f can be written as

$$f = g \circ q_{\pi/2},$$

In this case, f can also be written as

$$f = q_{\pi/2} \circ g.$$

where g is the scaling represented by $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $q_{\pi/2}$ is reflection in the y-axis.

The geometric effect of f is shown in Figure 3.6.

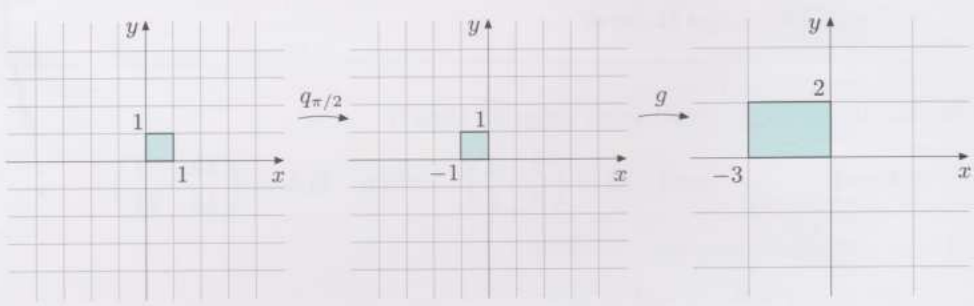


Figure 3.6 Geometric effect of f

Thus the reflection evident in Figure S.14 has been explained.

The scaling in Activity 2.4(c) is the subject of the following activity.

Activity 3.5 Interpreting a scaling

Let f be the scaling represented by the matrix

$$\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}.$$

(a) Show that

$$\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right].$$

Hence express f as a composite involving three linear transformations. Identify those transformations.

(b) Use equation (3.1) to express f as a composite involving just two linear transformations. Sketch the geometric effect of f.

Solutions are given on page 62.

In Section 5, you will use the computer to express other types of linear transformations as composites. In particular, you will see examples of linear transformations represented by matrices of the forms

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 \\ c & d \end{pmatrix},$$

where a, b, c and d are non-zero. Special cases of such linear transformations are the x-shears and y-shears you met in Subsection 2.2.

A 2×2 matrix that has a zero in the bottom left-hand corner or in the top right-hand corner is called a **triangular matrix**.

3.2 Inverse transformations

In this subsection, we explore whether it is possible to 'undo the effect' of a linear transformation by applying a second linear transformation. In Chapter A3, you saw that it is possible to undo the effect of a rotation r_θ by applying a second rotation $r_{-\theta}$. And a reflection q_θ is self-inverse, so its effect can be undone by applying q_θ again.

By contrast, a flattening of the plane cannot be undone. This is because a flattening is a *many-one* function, and it is impossible for a second function to send a point (such as the origin) back to more than one point.

But what about linear transformations that do not flatten the plane – can they *always* be undone? To help answer this question, recall that a matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the property that $\det \mathbf{A} \neq 0$, has an inverse matrix given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

Inverse matrices were discussed in MST121 Chapter B2, Section 5.

that satisfies

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}, \tag{3.4}$$

where $\mathbf{I}$ is the identity matrix.

Activity 3.6 Inverse of a matrix

Show that the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 8 \\ 4 & 6 \end{pmatrix}$$

has an inverse $\mathbf{A}^{-1}$, and hence determine it. Verify that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

A solution is given on page 62.

Now suppose that f is a linear transformation represented by a matrix $\mathbf{A}$ with inverse $\mathbf{A}^{-1}$. Then we can define the linear transformation

$$g(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x}.$$

For any $\mathbf{x} \in \mathbb{R}^2$, we have

$$(g \circ f)(\mathbf{x}) = g(f(\mathbf{x})) = g(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x},$$

so g undoes the effect of f, as required. Also

$$(f \circ g)(\mathbf{x}) = f(g(\mathbf{x})) = f(\mathbf{A}^{-1}\mathbf{x}) = \mathbf{A}(\mathbf{A}^{-1}\mathbf{x}) = (\mathbf{A}\mathbf{A}^{-1})\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x},$$

so f undoes the effect of g. The transformation g is called the *inverse of* f and it is denoted by f^{-1}.

The existence of this inverse f^{-1} has two important consequences, namely that f is both one-one and onto.

By contrast, recall from Subsection 2.3 that a flattening is neither one-one nor onto.

First, we show that f is a one-one function. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are vectors such that $f(\mathbf{x}) = f(\mathbf{y})$. Then

$$f^{-1}(f(\mathbf{x})) = f^{-1}(f(\mathbf{y})),$$

so $\mathbf{x} = \mathbf{y}$. Hence f is one-one.

Secondly, we show that f is an onto function. Let $\mathbf{y}$ be an arbitrary point of the codomain $\mathbb{R}^2$. Then

$$\mathbf{x} = f^{-1}(\mathbf{y}) \text{ lies in the domain } \mathbb{R}^2 \text{ of } f.$$

So

$$f(\mathbf{x}) = f(f^{-1}(\mathbf{y})) = \mathbf{y}.$$

Since $\mathbf{y}$ is an arbitrary point of the codomain $\mathbb{R}^2$, we conclude that $f(\mathbb{R}^2) = \mathbb{R}^2$. Hence f is onto.

We have therefore established the following general result.

Remember that $\mathbf{A}$ fails to be invertible if and only if $\det \mathbf{A} = 0$.

Inverse of a linear transformation

Let f be a linear transformation of the plane represented by a matrix $\mathbf{A}$.

- ◇ If $\mathbf{A}$ is invertible, then f is a one-one, onto function with inverse f^{-1}. This inverse is a linear transformation represented by the matrix $\mathbf{A}^{-1}$. We say that f is **invertible** (or **non-singular**).
- ◇ If the matrix $\mathbf{A}$ is not invertible, then f is a many-one function that flattens the plane. Therefore f has no inverse.

Example 3.1 Finding an inverse transformation

Let f be a linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}.$$

(a) Show that f is one-one and onto.

(b) Determine f^{-1}.

(c) Find the point (x, y) such that $f(x, y) = (2, 1)$.

Solution

(a) In this case,

$$\det \mathbf{A} = 3 \times 4 - 2 \times 5 = 2.$$

Since this determinant is non-zero, the matrix $\mathbf{A}$ is invertible. It follows that f is one-one and onto.

(b) The inverse f^{-1} is the linear transformation represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix}.$$

(c) The required point is $f^{-1}(2,1)$ with position vector

$$\mathbf{A}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{7}{2} \end{pmatrix}.$$

In the next activity, you are asked to investigate the geometric effect of the inverse of a linear transformation.

Activity 3.7 Finding an inverse transformation

Let f be a linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 9 & 5 \end{pmatrix}.$$

(a) Show that f is one-one and onto, and determine f^{-1}.

(b) Find the image of the unit square S under f, and check that $f(S)$ is sent back to S by f^{-1}.

(c) What are the factors by which f and f^{-1} scale areas, and what is the relationship between these factors?

Solutions are given on page 63.

The solution to this activity illustrates a general result, namely that the factor by which f^{-1} scales areas is the reciprocal of the factor by which f scales areas. In fact, we can use the result about the product of determinants on page 39 to express this observation in terms of determinants. For if f is represented by the matrix $\mathbf{A}$, then f^{-1} is represented by the matrix $\mathbf{A}^{-1}$. So we have

$$\det \mathbf{A} \det(\mathbf{A}^{-1}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det \mathbf{I} = 1.$$

This shows that $\det(\mathbf{A}^{-1})$ is the reciprocal of $\det \mathbf{A}$.

Determinant of an inverse matrix

If $\mathbf{A}$ is an invertible matrix, then $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det \mathbf{A}}$.

One rather surprising application of an inverse transformation f^{-1} is that it can be used to find the images of curves under f. The next example illustrates this by finding the image of the unit circle $x^2 + y^2 = 1$ under a particular linear transformation.

The same technique can be used to find the images of other curves.

Example 3.2 Area of an image

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & -3 \end{pmatrix}.$$

Find the equation of the image $f(\mathscr{C})$ of the unit circle $\mathscr{C}$ under f, and calculate the area enclosed by $f(\mathscr{C})$.

Solution

First observe that $\det \mathbf{A} = 1 \times (-3) - (-2) \times 1 = -1$, so f has an inverse transformation f^{-1} represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} -3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}.$$

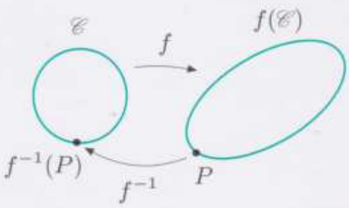


Figure 3.7 The points P and $f^{-1}(P)$

If P is an arbitrary point (x, y) on the image $f(\mathscr{C})$, then P must be the image under f of the point $f^{-1}(P)$ on $\mathscr{C}$; see Figure 3.7. The position vector of $f^{-1}(P)$ is

$$\mathbf{A}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - 2y \\ x - y \end{pmatrix}.$$

Since these components are the coordinates of a point on the unit circle $\mathscr{C}$, it follows that

$$(3x - 2y)^2 + (x - y)^2 = 1,$$

or equivalently

$$9x^2 - 12xy + 4y^2 + x^2 - 2xy + y^2 = 1;$$

that is

$$10x^2 - 14xy + 5y^2 = 1.$$

Using the techniques in Chapter A3, Section 4, we can show that this is the equation of an ellipse.

This is therefore the equation (satisfied by the points) of $f(\mathscr{C})$.

Remember that the area of a circle of radius r is πr^2.

Now the area enclosed by $\mathscr{C}$ is π. Since f scales areas by the factor $|\det \mathbf{A}| = |-1| = 1$, it follows that the area of $f(\mathscr{C})$ is also π.

The next activity provides practice in calculating such an area.

Activity 3.8 Area of an image

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Find the equation of the image $f(\mathscr{C})$ of the unit circle $\mathscr{C}$ under f, and calculate the area enclosed by $f(\mathscr{C})$.

A solution is given on page 63.

Now suppose that you are given the equation of an ellipse E. Provided that you can find a linear transformation f that maps the unit circle onto E, then you will be able to calculate the area enclosed by E.

Activity 3.9 Finding the area of an ellipse

(a) Give a geometric description of a linear transformation f that sends the unit circle $\mathscr{C}$ onto the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$. Hence write down the matrix that represents f, and use it to find the area enclosed by the ellipse.

(b) Show that the area of an ellipse of the form $x^2/a^2 + y^2/b^2 = 1$, where $a > 0$ and $b > 0$, is πab.

Solutions are given on page 63.

Summary of Section 3

This section has introduced:

◇ the composite of one linear transformation followed by another;

◇ the inverse of a linear transformation;

◇ the use of matrices to calculate composite and inverse transformations;

◇ the formulas $\det(\mathbf{BA}) = \det \mathbf{B} \det \mathbf{A}$ and $\det(\mathbf{A}^{-1}) = 1/\det \mathbf{A}$;

◇ the condition $\det \mathbf{A} \neq 0$ for a linear transformation $f(\mathbf{x}) = \mathbf{Ax}$ to have an inverse;

◇ the use of an inverse transformation to find the image of the unit circle, and hence the area enclosed by the (elliptical) image.

Exercises for Section 3

Exercise 3.1

Let f and g be linear transformations represented by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix},$$

respectively. Determine the matrices that represent each of the following composite transformations.

(a) $g \circ f$ (b) $f \circ g$

Exercise 3.2

For each of the following matrices $\mathbf{A}$ decide whether it is invertible. For those that are invertible, calculate $\mathbf{A}^{-1}$.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -1 \\ -8 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

Exercise 3.3

Let f be the linear transformation that maps $(1, 0)$ to $(2, 3)$ and $(0, 1)$ to $(-3, -4)$. Also let g be the linear transformation that maps $(1, 0)$ to $(1, 2)$ and $(0, 1)$ to $(-1, 1)$.

(a) Write down the matrices $\mathbf{A}$ and $\mathbf{B}$ that represent f and g, respectively.

(b) Use the matrix that represents f to find a linear transformation that maps $(2, 3)$ back to $(1, 0)$ and $(-3, -4)$ back to $(0, 1)$.

(c) Find a linear transformation that maps $(2, 3)$ to $(1, 2)$ and $(-3, -4)$ to $(-1, 1)$.

Exercise 3.4

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}.$$

Find the equation of the image $f(\mathscr{C})$ of the unit circle $\mathscr{C}$ under f, and calculate the area enclosed by $f(\mathscr{C})$.

4 Affine transformations

So far we have concentrated on linear transformations. These transformations map parallel lines onto parallel lines, and therefore have a homogeneous effect on the plane. As you have seen, this effect can be visualised by examining what happens to the unit grid.

See Figure 2.4, for example.

Linear transformations suffer from one serious limitation, namely that they leave the origin fixed. So, although we can use linear transformations to describe rotations about the origin, we cannot use them to describe rotations about other points. Nor can we use them to describe reflections in lines that do not pass through the origin. There are many other transformations of the plane that map parallel lines to parallel lines, and yet cannot be described by linear transformations. To overcome these limitations, *affine transformations* are now introduced.

The adjective 'affine' has the same root as the noun 'affinity'. Its use here signifies the existence of a relationship or resemblance between a figure and its image under such a transformation which, except in special cases, is more general than those of congruence and similarity.

4.1 Definition of an affine transformation

The effect on the unit grid of any transformation that maps parallel lines to parallel lines can be determined from its effect on just three points, namely $(0,0)$, $(1,0)$, $(0,1)$. The top parts of Figure 4.1 illustrate this for a transformation f that sends the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $A(2,1)$, $B(5,3)$, $C(3,4)$, respectively. Because f preserves parallelism, it must send the unit square to the parallelogram $ABDC$, and this determines the effect that f has on the entire unit grid.

D is the image of $(1,1)$ under f.

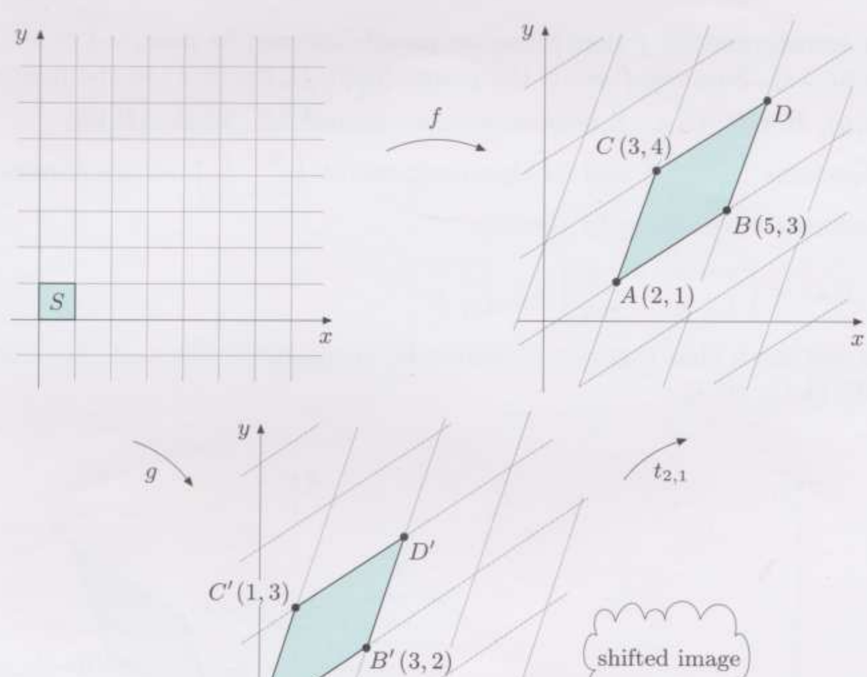


Figure 4.1 Decomposing the transformation f

How can this effect be described algebraically? Suppose, for a moment, that the *image* of the unit grid is shifted 1 unit down, and 2 units to the

left of its actual position, as shown at the bottom of Figure 4.1. In this 'shifted image', A has moved to the origin, B has moved to B' with coordinates $(5-2, 3-1) = (3, 2)$ and C has moved to C' with coordinates $(3-2, 4-1) = (1, 3)$. We can obtain the shifted image from the unit grid by applying the linear transformation g represented by the matrix

Remember that the first column of $\mathbf{A}$ is the position vector of B', the image of $(1, 0)$ under g, and the second column is the position vector of C', the image of $(0, 1)$ under g (see the boxed result on page 19).

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}.$$

We can then map the shifted image grid to its actual position by applying the translation $t_{2,1}$. Thus f is equal to the composite function $t_{2,1} \circ g$. So the effect of f on an arbitrary point with position vector $\mathbf{x}$ is described by the rule

$$f(\mathbf{x}) = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A quick way to write down such a rule is to note that the first column of the matrix $\mathbf{A}$ is the vector $\overrightarrow{AB}$, the second column is $\overrightarrow{AC}$, and the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is the position vector of A. Use this method in the following activity.

Activity 4.1 Finding a transformation that preserves parallelism

Write down a transformation f that preserves parallelism and sends the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $A = (5, 3)$, $B = (7, 4)$, $C = (8, 6)$, respectively.

A solution is given on page 64.

Any transformation f that preserves parallelism can be analysed in a similar way. Suppose f sends the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $A(p, q)$, $B(s, t)$, $C(u, v)$, respectively; see Figure 4.2. Then $\overrightarrow{AB}$ has components $\begin{pmatrix} s-p \\ t-q \end{pmatrix}$ and $\overrightarrow{AC}$ has components $\begin{pmatrix} u-p \\ v-q \end{pmatrix}$, so the required transformation is given by the rule

$$f(\mathbf{x}) = \begin{pmatrix} s-p & u-p \\ t-q & v-q \end{pmatrix}\mathbf{x} + \begin{pmatrix} p \\ q \end{pmatrix}.$$

You can check that this rule is correct by checking its effect on the points $(0, 0)$, $(1, 0)$, $(0, 1)$.

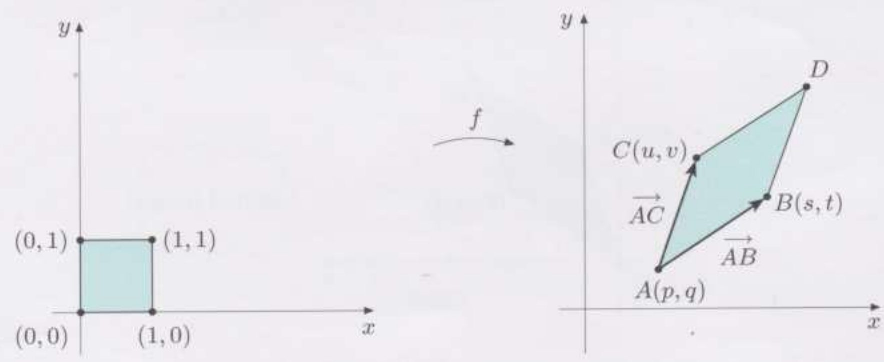


Figure 4.2 Defining the transformation f

So you now have a way of finding a function of the form $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ that sends $(0,0)$, $(1,0)$, $(0,1)$ to any three given points. Functions that have this form are known as *affine* transformations.

An **affine transformation** of the plane is a function of the form

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\mathbf{x} \longmapsto \mathbf{A}\mathbf{x} + \mathbf{a},$$

where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a vector with two components.

As with linear transformations, it is possible for an affine transformation to flatten the plane. This occurs when one of the vectors $\overrightarrow{AB}$ or $\overrightarrow{AC}$ in Figure 4.2 is a scalar multiple of the other. In such a case, the matrix $\mathbf{A}$ is not invertible.

Some texts reserve the term 'affine' for transformations where the matrix $\mathbf{A}$ in the rule $\mathbf{x} \longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}$ is invertible.

For ease of reference, the method for finding an affine transformation is summarised in the following boxed result.

Determining an affine transformation

The affine transformation $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that maps the points $(0,0)$, $(1,0)$, $(0,1)$ to the points (p,q), (s,t), (u,v), respectively, is given by

$$f(\mathbf{x}) = \begin{pmatrix} s-p & u-p \\ t-q & v-q \end{pmatrix}\mathbf{x} + \begin{pmatrix} p \\ q \end{pmatrix}.$$

In Activity 2.6, you saw how a linear transformation can be used to find the area of a triangle that has a vertex at the origin. The same approach can now be used to find the area of any triangle.

Example 4.1 Area of a triangle

(a) Write down the affine transformation that sends the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $(2,5)$, $(3,-7)$, $(1,-3)$, respectively.

(b) Hence find the area of the triangle T with vertices at $(2,5)$, $(3,-7)$, $(1,-3)$.

Solution

(a) The required affine transformation f has the rule $\mathbf{x} \longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}$, where

$$\mathbf{A} = \begin{pmatrix} 3-2 & 1-2 \\ -7-5 & -3-5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -12 & -8 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

(b) Since the translation through the vector $\mathbf{a}$ has no effect on areas, it follows that the affine transformation scales areas by the factor

$$|\det \mathbf{A}| = |1 \times (-8) - (-1) \times (-12)| = 20.$$

The triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ has area $\frac{1}{2}$, so the area of T must be $20 \times \frac{1}{2} = 10$.

The important thing to notice about this example is that the translation through the vector $\mathbf{a}$ has no effect on areas, so the factor by which an affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ scales areas is simply $|\det \mathbf{A}|$.

The translation also has no effect on orientation, so f preserves orientation if $\det \mathbf{A}$ is positive, and reverses orientation if $\det \mathbf{A}$ is negative.

Activity 4.2 Finding the area of a triangle

(a) Write down the affine transformation that sends the points $(0,0)$, $(1,0)$, $(0,1)$ to the points $(4,-3)$, $(-1,5)$, $(3,7)$, respectively.

(b) Hence find the area of the triangle T with vertices at $(4,-3)$, $(-1,5)$, $(3,7)$.

Solutions are given on page 64.

4.2 General rotations and reflections

So far, we have confined our attention to those rotations and reflections that fix the origin. But with the introduction of affine transformations, this constraint of a fixed origin no longer applies. Suppose we wish to investigate the transformation that rotates the plane about the point $(2,3)$ through an (anticlockwise) angle $\pi/2$. How can this transformation be described algebraically?

You have already seen that an (anticlockwise) rotation through $\pi/2$ about the origin can be expressed algebraically by the linear transformation $r_{\pi/2}$, which is represented by the matrix

See line above Activity 1.6.

$$\mathbf{R}_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One approach to achieving the rotation about $(2,3)$ (see Figure 4.3) is first to translate the plane so that the point $(2,3)$ ends up at the origin. The translated plane can then be rotated about the origin through $\pi/2$, and finally it can be returned to its correct position by applying the translation that sends the origin back to the point $(2,3)$.

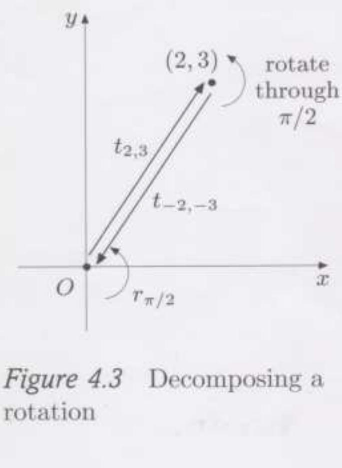


Figure 4.3 Decomposing a rotation

Algebraically, this is equivalent to the composite of three transformations, namely $t_{2,3} \circ (r_{\pi/2} \circ t_{-2,-3})$. Under this composite transformation, an arbitrary point $\mathbf{x}$ is mapped to

$$\begin{aligned}
\mathbf{R}_{\pi/2}\left(\mathbf{x} + \begin{pmatrix} -2 \\ -3 \end{pmatrix}\right) + \begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}.
\end{aligned}$$

So the rotation through $\pi/2$ about the point $(2,3)$ can be expressed as the affine transformation $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

A check on our working is to show that the centre of rotation $(2,3)$ remains fixed. Here

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

as required.

In the following activity, you are asked to produce an argument like that above in order to find another rotation.

Activity 4.3 Finding a rotation

Find the affine transformation that describes an anticlockwise rotation about the point $(3, 1)$ through $\pi/3$.

A solution is given on page 64.

The idea of performing 'the same action (a rotation) somewhere else' can also be applied to reflections. The next activity asks you to find an algebraic description for a reflection in a line that does not pass through the origin. In this case, you can choose to translate any point on the axis of reflection to the origin and use your knowledge of the matrix for a reflection in a line through the origin.

Activity 4.4 Finding a reflection

(a) Determine the matrix that represents reflection in the line $y = x$.

(b) By first translating the point $(0, 4)$ to the origin, find the affine transformation that describes reflection in the line $y = x + 4$.

Solutions are given on page 64.

Summary of Section 4

This section has introduced:

◇ the definition of an affine transformation, and its interpretation as a function that preserves parallelism;

◇ a technique for finding the area of a triangle given the coordinates of its vertices;

◇ a technique for finding the affine transformations representing general rotations and reflections.

Exercises for Section 4

Exercise 4.1

(a) Write down the affine transformation that sends the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(4, -2)$, $(10, 5)$, $(8, 7)$, respectively.

(b) Hence find the area of the triangle T with vertices at $(4, -2)$, $(10, 5)$, $(8, 7)$.

Exercise 4.2

Find the affine transformation that describes an anticlockwise rotation about the point $(4, 6)$ through $\pi/4$ radians.

5 Visualising affine transformations

In this section you will need computer access and Computer Book B.

The computer can be used to illustrate the effect of linear transformations on the unit grid and polygonal figures. This makes it possible to check your understanding of the way scalings, shears and flattenings behave. It can also help you to explore the behaviour of less familiar linear transformations, and to investigate ways of decomposing them into a composite of simpler linear transformations. In this connection, rotations about the origin, reflections in lines through the origin, scalings, x-shears, y-shears and flattenings are referred to as **basic linear transformations**.

Affine transformations can be investigated in a similar way. In particular, you are invited to explore ways of sending one figure onto another.

Refer to Computer Book B for the work in this section.

Summary of Section 5

This section has used the computer to explore the behaviour of various linear and affine transformations.

Summary of Chapter B2

In this chapter you saw how certain isometries of the plane can be written in matrix form. This led to the idea of a linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ represented by a matrix $\mathbf{A}$. You saw how the unit grid can be used to explore the various kinds of behaviour exhibited by a linear transformation – in particular, scalings, uniform scalings, shears and flattenings.

One striking feature of a linear transformation is that it has a homogeneous effect on the plane. In particular, the factor by which the area of a figure is scaled is independent of the figure's location in the plane. In fact, the scale factor for the area is equal to $|\det \mathbf{A}|$ and this enables us to use determinants to calculate the area of certain figures like triangles and ellipses.

Finally you met the idea of an affine transformation. This combines translations with linear transformations, allowing us to study figures located anywhere in the plane.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Translation through a vector, position vector, linear transformation, matrix representation of a linear transformation, identity transformation, unit grid, unit square, scaling, uniform scaling, diagonal matrix, shear, flattening, zero transformation, orientation of a figure, one-one function, many-one function, onto function, triangular matrix, invertible linear transformation, affine transformation.

Notation to know and use

- ◇ $\overrightarrow{AB}$ for the vector represented by an arrow from A to B.
- ◇ $\mathbf{p}$ for the position vector of the point P.
- ◇ $\mathbf{R}_\theta$ and $\mathbf{Q}_\theta$ for the matrices that represent r_θ and q_θ, respectively.
- ◇ $\det \mathbf{A}$ for the determinant of a matrix $\mathbf{A}$.

Mathematical skills

- ◇ Use vector algebra to express the position vector of a point in terms of the position vectors of other points.
- ◇ Use the matrix representations of rotations (about the origin), reflections (in lines through the origin), scalings, shears and flattenings.
- ◇ Write down the matrix representing a linear transformation given the images of the points with coordinates $(1, 0)$ and $(0, 1)$.
- ◇ Find the image of a polygonal figure under a linear transformation.
- ◇ Compose transformations by using matrix multiplication.

- ◇ Find the inverse of a transformation represented by an invertible matrix.
- ◇ Use the inverse of a linear transformation f to find the image of the unit circle under f.
- ◇ Use determinants to calculate the area of the image of a figure under a linear or an affine transformation.
- ◇ Determine the affine transformation that represents a general rotation or reflection.

Ideas to be aware of

- ◇ Be aware of the type of rules that give rise to linear transformations.
- ◇ Know that linear transformations have a 'homogeneous' effect on the plane.
- ◇ Know the various (equivalent) conditions on a matrix, or linear transformation, that give rise to flattenings, namely: matrix has zero determinant, one column of matrix is a scalar multiple of the other; matrix is not invertible; transformation is not invertible; transformation is not onto; transformation is many-one.

Mathcad skills

- ◇ Investigate linear transformations by examining their effect on the unit grid.
- ◇ Investigate images under linear and affine transformations.

Solutions to Activities

Solution 1.1

In each of parts (a)–(d), the x- and y-axes are to be taken to be across and up the page, respectively.

(a)

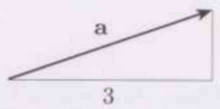


Figure S.1

(b)

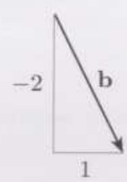


Figure S.2

(c)

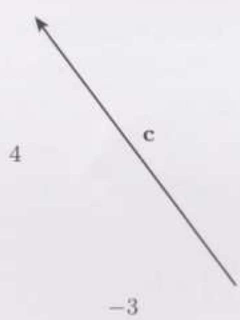


Figure S.3

(d)

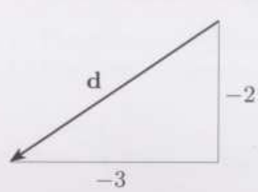


Figure S.4

Solution 1.2

(a) Here

$$\mathbf{c} = 4\mathbf{a} = 4\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}16\\12\end{pmatrix}.$$

(b) We have

$$\mathbf{c} = -2\mathbf{b} = -2\begin{pmatrix}-5\\2\end{pmatrix} = \begin{pmatrix}10\\-4\end{pmatrix}.$$

(c) Here

$$\begin{aligned}\mathbf{c} &= 4\mathbf{a} - 2\mathbf{b}\\ &= 4\begin{pmatrix}4\\3\end{pmatrix} - 2\begin{pmatrix}-5\\2\end{pmatrix}\\ &= \begin{pmatrix}16+10\\12-4\end{pmatrix}\\ &= \begin{pmatrix}26\\8\end{pmatrix}.\end{aligned}$$

(d) In this case

$$\begin{aligned}\mathbf{c} &= 2\mathbf{a} + \tfrac{3}{2}\mathbf{b}\\ &= 2\begin{pmatrix}4\\3\end{pmatrix} + \tfrac{3}{2}\begin{pmatrix}-5\\2\end{pmatrix}\\ &= \begin{pmatrix}8-\frac{15}{2}\\6+3\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{2}\\9\end{pmatrix}.\end{aligned}$$

Solution 1.3

In each of parts (a)–(d), the x- and y-axes are to be taken to be across and up the page, respectively.

(a) Here, $\mathbf{c} = 4\mathbf{a}$ is the vector that has 4 times the magnitude of $\mathbf{a}$ and the same direction.

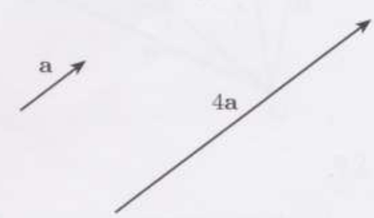


Figure S.5

(b) In this case, $\mathbf{c} = -2\mathbf{b}$ is the vector that has twice the magnitude of $\mathbf{b}$, but points in the opposite direction.

Figure S.6

(c) Here, $\mathbf{c} = 4\mathbf{a} - 2\mathbf{b}$ is represented by the arrow from the tail of the arrow for $4\mathbf{a}$ to the tip of the arrow for $-2\mathbf{b}$.

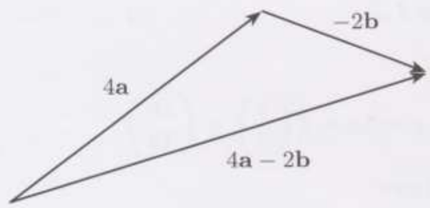


Figure S.7

(d) In this case, $\mathbf{c} = 2\mathbf{a} + \frac{3}{2}\mathbf{b}$ is represented by the arrow from the tail of the arrow for $2\mathbf{a}$ to the tip of the arrow for $\frac{3}{2}\mathbf{b}$.

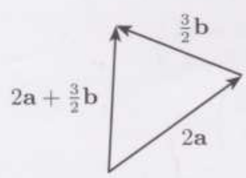


Figure S.8

Solution 1.4

(a) Using the arrows that link Q to R via P, we have

$$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR} = -\overrightarrow{SR} + \overrightarrow{PR} = \overrightarrow{PR} - \overrightarrow{SR}.$$

(b) Here

$$\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PR} - \overrightarrow{PQ}.$$

Solution 1.5

(a)

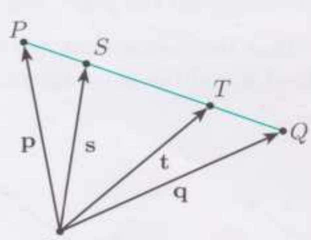


Figure S.9

First let $\mathbf{s}$ be the position vector of S. Then

$$\mathbf{s} = \overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = \mathbf{p} + \tfrac{1}{4}\overrightarrow{PQ}.$$

But $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$, so

$$\mathbf{s} = \mathbf{p} + \tfrac{1}{4}(\mathbf{q} - \mathbf{p}) = \tfrac{3}{4}\mathbf{p} + \tfrac{1}{4}\mathbf{q}.$$

Next let $\mathbf{t}$ be the position vector of T. Then

$$\mathbf{t} = \overrightarrow{OT} = \overrightarrow{OP} + \overrightarrow{PT} = \mathbf{p} + \tfrac{2}{3}\overrightarrow{PQ},$$

so

$$\mathbf{t} = \mathbf{p} + \tfrac{2}{3}(\mathbf{q} - \mathbf{p}) = \tfrac{1}{3}\mathbf{p} + \tfrac{2}{3}\mathbf{q}.$$

(b) In the particular case where $P = (2, 5)$ and $Q = (1, -3)$, we have

$$\mathbf{s} = \tfrac{3}{4}\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ 3 \end{pmatrix}$$

and

$$\mathbf{t} = \tfrac{1}{3}\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Solution 1.6

In each case, we obtain the required matrix by replacing θ in the general form for $\mathbf{R}_\theta$ by the specific angle of rotation.

Since $\cos 0 = 1$ and $\sin 0 = 0$, r_0 is represented by

$$\mathbf{R}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(This is the 2×2 identity matrix, which we usually write as $\mathbf{I}$.)

Since $\cos\pi = -1$ and $\sin\pi = 0$, r_π is represented by

$$\mathbf{R}_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $\cos(3\pi/2) = 0$ and $\sin(3\pi/2) = -1$, $r_{3\pi/2}$ is represented by

$$\mathbf{R}_{3\pi/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$, $r_{2\pi}$ is represented by

$$\mathbf{R}_{2\pi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Solution 1.7

The rotation $r_{\pi/4}$ is represented by the matrix

$$\mathbf{R}_{\pi/4} = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

The image of the unit square therefore has vertices given by

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

The image of S is the square with vertices at $(0,0)$, $(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, $(0, \sqrt{2})$ and $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$. This image is shown below.

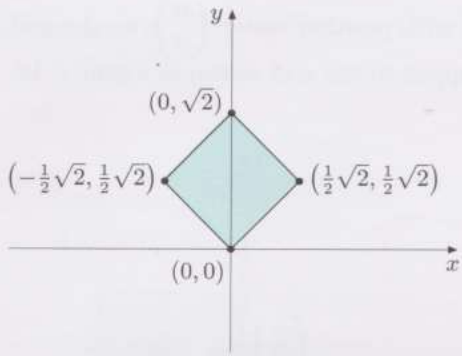


Figure S.10

Solution 1.8

The reflection $q_{\pi/4}$ is represented by the matrix

$$\mathbf{Q}_{\pi/4} = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ \sin(\pi/2) & -\cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The image of the unit square therefore has vertices given by:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The image is the square with vertices at $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. So, although two of the vertices swap over, the image of the unit square is itself.

Solution 2.1

(a) f is the linear transformation represented by the matrix

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

(To ensure the correct ordering of the matrix elements, you may find it helps to rewrite the image $(x+2y, y-x)$ in the form $(x+2y, -x+y)$.)

(b) f is not a linear transformation because it maps $(0,0)$ to $(2,0)$.

(c) f is the linear transformation represented by the zero matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(d) f is not a linear transformation because it maps $(0,0)$ to $(2,-1)$.

(e) f is not a linear transformation because it reflects $(0,0)$ to $(4,0)$.

(f) f is the linear transformation represented by the matrix

$$\begin{pmatrix} \cos(-\frac{1}{4}\pi) & -\sin(-\frac{1}{4}\pi) \\ \sin(-\frac{1}{4}\pi) & \cos(-\frac{1}{4}\pi) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}.$$

Solution 2.2

(a) The vertices of $g(S)$ have position vectors:

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix};$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix};$$
$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Since f maps lines to lines, we deduce that the image of S is the parallelogram shown in Figure S.11.

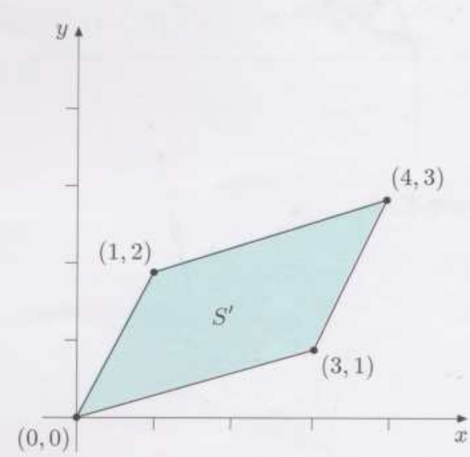


Figure S.11

(b) The vertex (with position vector) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is the first column of $\mathbf{A}$.

Similarly, the vertex (with position vector) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the second column of $\mathbf{A}$.

Finally, the vertex $(4,3)$ has position vector

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

the sum of the columns of $\mathbf{A}$.

Solution 2.3

(a) From equation (2.2) we know that $f(1,2)$ has position vector $\mathbf{a}+2\mathbf{b}$. We can therefore plot its position by starting at the origin and counting one parallelogram 'length' in the direction of $\mathbf{a}$, followed by two parallelogram 'lengths' in the direction of $\mathbf{b}$ (see Figure S.12).

(b) Here $f(2,-1)$ has position vector $2\mathbf{a}-\mathbf{b}$, so we can plot its position by counting 2 parallelogram 'lengths' from the origin in the direction of $\mathbf{a}$ followed by 1 parallelogram 'length' in the opposite direction to $\mathbf{b}$.

(c) Finally $f(0,2)$ has position vector $2\mathbf{b}$, so we can plot its position simply by counting 2 parallelogram 'lengths' from the origin in the direction of $\mathbf{b}$.

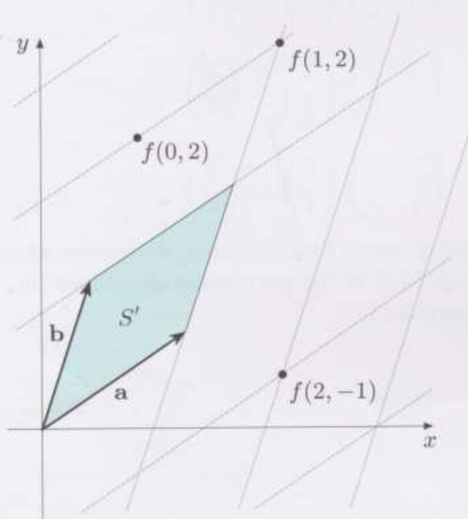


Figure S.12

Solution 2.4

(a) In this case f maps $(1,0)$ to the point with position vector $\begin{pmatrix}3\\0\end{pmatrix}$, and it maps $(0,1)$ to the point with position vector $\begin{pmatrix}0\\2\end{pmatrix}$, so the unit grid is mapped to the grid shown in Figure S.13.

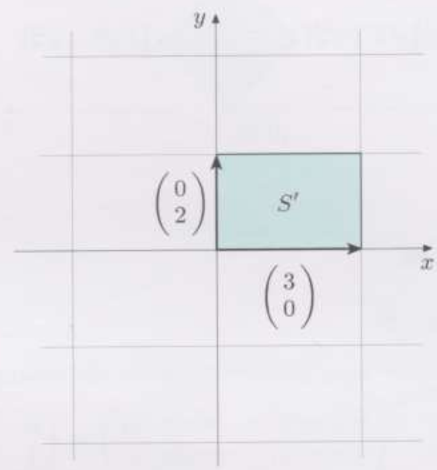


Figure S.13

The image grid is not skewed, but distances parallel to the x-axis are scaled by the factor 3, and distances parallel to the y-axis are scaled by the factor 2.

Since the unit square S is mapped to the rectangle S' with vertices at $(0,0)$, $(3,0)$, $(3,2)$, $(0,2)$, it follows that areas are increased by the factor $3\times 2=6$. This factor is just the product of the diagonal elements of $\mathbf{A}$.

(b) Here f maps $(1,0)$ to the point with position vector $\begin{pmatrix}-3\\0\end{pmatrix}$, and it maps $(0,1)$ to the point with position vector $\begin{pmatrix}0\\2\end{pmatrix}$, so the unit grid is mapped to the grid shown in Figure S.14.

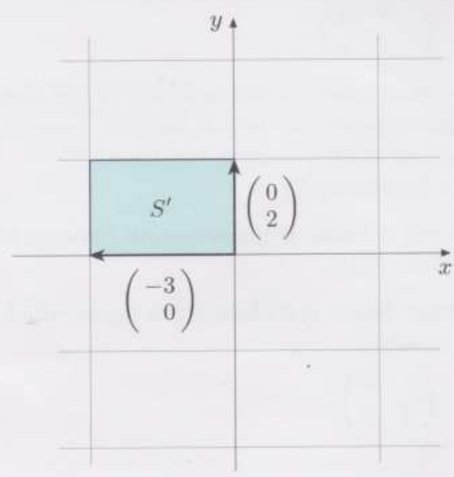


Figure S.14

Once again the image grid is not skewed. Distances parallel to the x-axis are scaled by the factor $|-3| = 3$, and distances parallel to the y-axis are scaled by the factor 2.

Since the unit square S is mapped to the rectangle S' with vertices at $(-3,0)$, $(0,0)$, $(0,2)$, $(-3,2)$, areas are again increased by the factor $3 \times 2 = 6$. However, in this case the factor is equal to the *modulus* of the product of the diagonal elements.

(c) Here f maps $(1,0)$ to the point with position vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$, and it maps $(0,1)$ to the point with position vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, so the unit grid is mapped to the grid shown in Figure S.15.

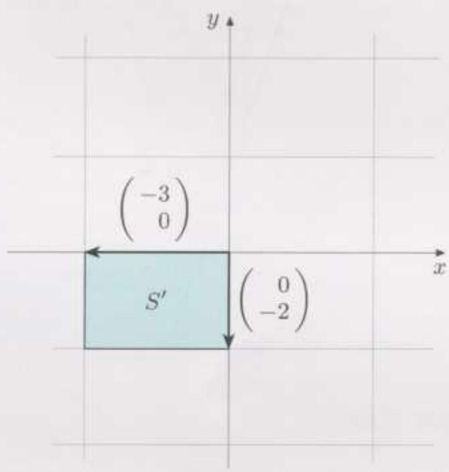


Figure S.15

As before the image grid is not skewed. Distances parallel to the x-axis are again scaled by the factor $|-3| = 3$, and distances parallel to the y-axis are scaled by the factor $|-2| = 2$.

Since the unit square S is mapped to the rectangle S' with vertices at $(-3,-2)$, $(0,-2)$, $(0,0)$, $(-3,0)$, it follows that areas are increased by the factor $3 \times 2 = 6$. This factor is equal to the product of the diagonal elements.

Solution 2.5

(a) In this case, f leaves $(1,0)$ unchanged, and it maps $(0,1)$ to the point with position vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the unit grid is mapped to the grid shown in Figure S.16.

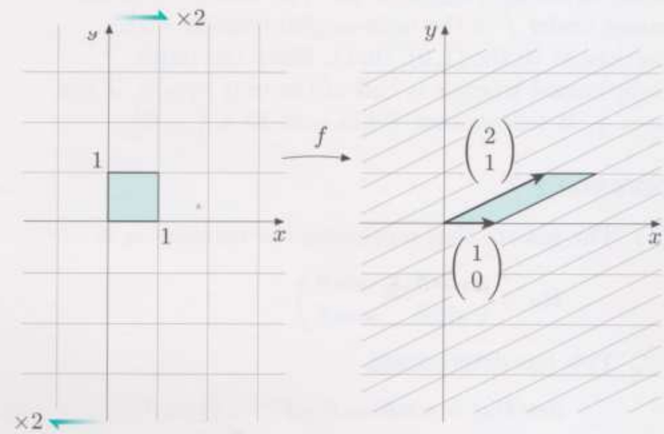


Figure S.16

The effect of the transformation is to *shear* the plane parallel to the x-axis, as indicated by the 'half' arrows. That is, each point moves parallel to the x-axis through a distance that is proportional to its height above the x-axis. Here height and distance are 'signed', so points below the axis move in the opposite direction to those above the axis. (The effect on the unit square is rather like a cross-section through a stack of paper resting on the x-axis. As the top sheet is pushed to the right each of the other sheets is pulled a distance proportional to its height in the stack. The resulting shape is a parallelogram.)

(b) Although the transformation changes the shape of the unit square into a parallelogram, the area of the parallelogram is the same as the area of the unit square. Indeed, by using the formula for the area of a parallelogram (see Figure 2.9), we find that its area is $1 \times 1 = 1$. It follows that the shear leaves areas unchanged.

Also, under the shear, the ordering of the unit square's vertices remains unchanged, so the shear preserves orientation, as shown in Figure S.17.

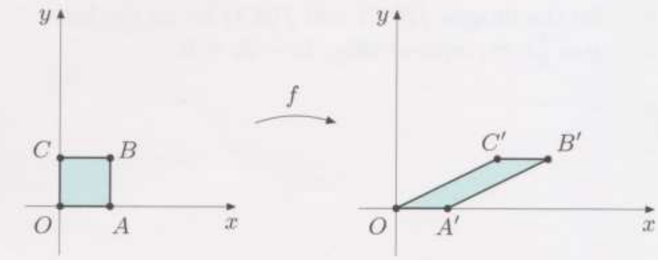


Figure S.17

Solution 2.6

Since f sends $(1,0)$ to $(2,5)$ and $(0,1)$ to $(3,1)$, it must be represented by the matrix

$$\begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}.$$

This has determinant $2 \times 1 - 3 \times 5 = -13$, so f scales areas by a factor of 13. The triangle T is the image under f of the right-angled triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$. Since the latter right-angled triangle is half of the unit square, it has area $\frac{1}{2}$. It follows that T has area $13 \times \frac{1}{2} = 6\frac{1}{2}$.

Solution 2.7

(a) The matrix that represents the rotation r_θ is

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This has determinant

$$\begin{aligned}\det(\mathbf{R}_\theta) &= \cos\theta\cos\theta - (-\sin\theta)\sin\theta \\ &= \cos^2\theta + \sin^2\theta \\ &= 1.\end{aligned}$$

This answer is to be expected since the rotation preserves both areas and orientation.

(b) The matrix that represents the reflection q_θ is

$$\mathbf{Q}_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

This has determinant

$$\begin{aligned}\det(\mathbf{Q}_\theta) &= \cos(2\theta)(-\cos(2\theta)) - \sin(2\theta)\sin(2\theta) \\ &= -\cos^2(2\theta) - \sin^2(2\theta) \\ &= -(\cos^2(2\theta) + \sin^2(2\theta)) \\ &= -1.\end{aligned}$$

Again, this answer is to be expected since the reflection preserves areas but reverses orientation.

Solution 2.8

(a) The position vectors of $f(1,0)$ and $f(0,1)$ are

$$\begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 6 & 2 \\ 9 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

So the images $f(1,0)$ and $f(0,1)$ lie on the line $y = \frac{3}{2}x$ or, equivalently, $3x - 2y = 0$.

(b) The image of (x,y) therefore has position vector

$$\begin{aligned}x\begin{pmatrix} 6 \\ 9 \end{pmatrix} + y\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= 3x\begin{pmatrix} 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= (3x+y)\begin{pmatrix} 2 \\ 3 \end{pmatrix}.\end{aligned}$$

(c) It follows from part (b) that each point on the line $3x + y = k$ maps to the point with position vector $k\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ on the line $y = \frac{3}{2}x$. In other words, the plane is flattened onto the line $3x - 2y = 0$.

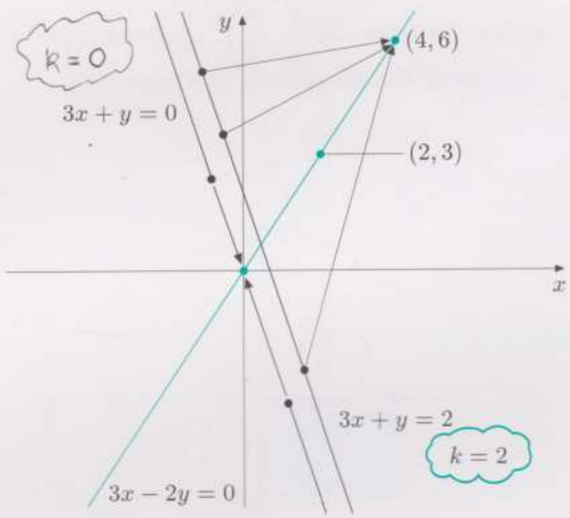


Figure S.18

(d) $\det \mathbf{A} = 6 \times 3 - 2 \times 9 = 0$.

Solution 2.9

(a) This matrix has the form

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

with $2\theta = \frac{1}{2}\pi$, so f is the reflection $q_{\pi/4}$ in the line $y = x$. (Of course, you might have recognised $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as the matrix of the reflection $q_{\pi/4}$, by interpreting its columns as giving the images of $(1,0)$ and $(0,1)$.)

In this case

$$\det \mathbf{A} = 0 \times 0 - 1 \times 1 = -1,$$

so, as expected, areas are scaled by the factor 1 (that is, figures remain the same size), and orientation is reversed.

(b) The first column of this matrix is $\frac{2}{3}$ of the second column, so this matrix represents a flattening. Indeed, the image of an arbitrary point (x, y) has position vector

$$x\begin{pmatrix}2\\4\end{pmatrix} + y\begin{pmatrix}3\\6\end{pmatrix} = (2x+3y)\begin{pmatrix}1\\2\end{pmatrix},$$

so f is a flattening onto the line $y - 2x = 0$. Areas are scaled by the factor

$$|\det \mathbf{A}| = |2 \times 6 - 3 \times 4| = 0,$$

confirming that f is a flattening.

(c) This matrix represents a scaling with factors 5 and 2. That is, the plane is scaled by the factor 5 parallel to the x-axis, and by the factor 2 parallel to the y-axis.

We have

$$\det \mathbf{A} = 5 \times 2 - 0 \times 0 = 10$$

(the product of the diagonal elements of $\mathbf{A}$). It follows that areas are scaled by the factor 10 and orientation is preserved.

Solution 2.10

(a) Suppose (r, s) and (u, v) are points such that $f(r, s) = f(u, v)$. Then

$$\begin{pmatrix}0 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}r\\s\end{pmatrix} = \begin{pmatrix}0 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix};$$

that is,

$$\begin{pmatrix}2s\\3r+4s\end{pmatrix} = \begin{pmatrix}2v\\3u+4v\end{pmatrix}.$$

Equating components, we obtain $2s = 2v$ and $3r + 4s = 3u + 4v$, from which we conclude that $s = v$, and hence $r = u$. It follows that $(r, s) = (u, v)$. Hence f is one-one.

(b) Let (u, v) be an arbitrary point in the codomain $\mathbb{R}^2$. For (u, v) to be the image of a point (x, y) in the domain, we require $f(x, y) = (u, v)$; that is,

$$\begin{pmatrix}0 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}u\\v\end{pmatrix};$$

or, equivalently,

$$\begin{pmatrix}2y\\3x+4y\end{pmatrix} = \begin{pmatrix}u\\v\end{pmatrix}.$$

Solving for (x, y), we obtain $y = \frac{1}{2}u$ and hence $x = \frac{1}{3}(v - 4y) = \frac{1}{3}(v - 2u)$. So one point that maps to (u, v) is $(x, y) = \left(\frac{1}{3}(v-2u), \frac{1}{2}u\right)$. Since (u, v) is an arbitrary point in the codomain $\mathbb{R}^2$, we conclude that $f(\mathbb{R}^2) = \mathbb{R}^2$. Hence f is onto.

Solution 3.1

(a) From the general form of a rotation matrix, we have:

$$\begin{aligned}\mathbf{R}_{\pi/3} &= \begin{pmatrix}\cos(\pi/3) & -\sin(\pi/3)\\ \sin(\pi/3) & \cos(\pi/3)\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix};\end{aligned}$$

$$\begin{aligned}\mathbf{R}_{\pi/6} &= \begin{pmatrix}\cos(\pi/6) & -\sin(\pi/6)\\ \sin(\pi/6) & \cos(\pi/6)\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{2}\sqrt{3} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\sqrt{3}\end{pmatrix};\end{aligned}$$

$$\begin{aligned}\mathbf{R}_{\pi/2} &= \begin{pmatrix}\cos(\pi/2) & -\sin(\pi/2)\\ \sin(\pi/2) & \cos(\pi/2)\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.\end{aligned}$$

(b) We have

$$\begin{aligned}\mathbf{R}_{\pi/6}\mathbf{R}_{\pi/3} &= \begin{pmatrix}\frac{1}{2}\sqrt{3} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\sqrt{3}\end{pmatrix}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{4}\sqrt{3} - \frac{1}{4}\sqrt{3} & -\frac{3}{4} - \frac{1}{4}\\ \frac{1}{4} + \frac{3}{4} & -\frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{3}\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\\ &= \mathbf{R}_{\pi/2}.\end{aligned}$$

(c) For these linear transformations, composition corresponds to matrix multiplication.

Solution 3.2

(a) (i) The composite $g \circ f$ is the linear transformation represented by the matrix

$$\mathbf{BA} = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}.$$

(ii) The composite $f \circ g$ is the linear transformation represented by the matrix

$$\mathbf{AB} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}.$$

(b) Figure 3.3 shows the effect of first doing f, the x-shear, then doing g. That is, it illustrates the transformation $g \circ f$.

Solution 3.3

(a) The composite $r_\theta \circ q_\phi$ is represented by the matrix

$$\begin{aligned}\mathbf{R}_\theta\mathbf{Q}_\phi &= \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos(2\phi) & \sin(2\phi)\\ \sin(2\phi) & -\cos(2\phi)\end{pmatrix}\\ &= \begin{pmatrix}\cos\theta\cos(2\phi)-\sin\theta\sin(2\phi) & \cos\theta\sin(2\phi)+\sin\theta\cos(2\phi)\\ \sin\theta\cos(2\phi)+\cos\theta\sin(2\phi) & \sin\theta\sin(2\phi)-\cos\theta\cos(2\phi)\end{pmatrix}\\ &= \begin{pmatrix}\cos(\theta+2\phi) & \sin(\theta+2\phi)\\ \sin(\theta+2\phi) & -\cos(\theta+2\phi)\end{pmatrix}.\end{aligned}$$

This matrix represents reflection in a line ℓ through the origin.

(b) If ℓ makes an angle α with the positive x-axis, then $\theta + 2\phi = 2\alpha$, so $\alpha = \frac{1}{2}\theta + \phi$.

Solution 3.4

Let f and g be represented by the matrices $\mathbf{A}$ and $\mathbf{B}$ respectively. Then $g \circ f$ is represented by the matrix $\mathbf{BA}$, and

$$\det(\mathbf{BA}) = \det\mathbf{B}\det\mathbf{A}.$$

(a) If g is a flattening, then $\det\mathbf{B} = 0$ and

$$\det(\mathbf{BA}) = 0 \times \det\mathbf{A} = 0.$$

So $g \circ f$ is a flattening.

(b) Conversely, if $g \circ f$ is a flattening, then

$$\det\mathbf{B}\det\mathbf{A} = \det(\mathbf{BA}) = 0;$$

so either $\det\mathbf{B} = 0$ or $\det\mathbf{A} = 0$ (or both). That is, either g is a flattening or f is a flattening (or both).

Solution 3.5

(a) We have

$$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix};$$

hence

$$\begin{aligned}&\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\left[\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\\ &= \begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}-3 & 0\\ 0 & -2\end{pmatrix},\end{aligned}$$

as required.

Thus the scaling f can be written as

$$f = g \circ (q_{\pi/2} \circ q_0),$$

where g is the scaling represented by the matrix $\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$, $q_{\pi/2}$ is reflection in the y-axis and q_0 is reflection in the x-axis.

(b) From equation (3.1), $q_{\pi/2} \circ q_0 = r_\pi$. Thus the scaling f can be written as

$$f = g \circ r_\pi.$$

The geometric effect of f is shown below.

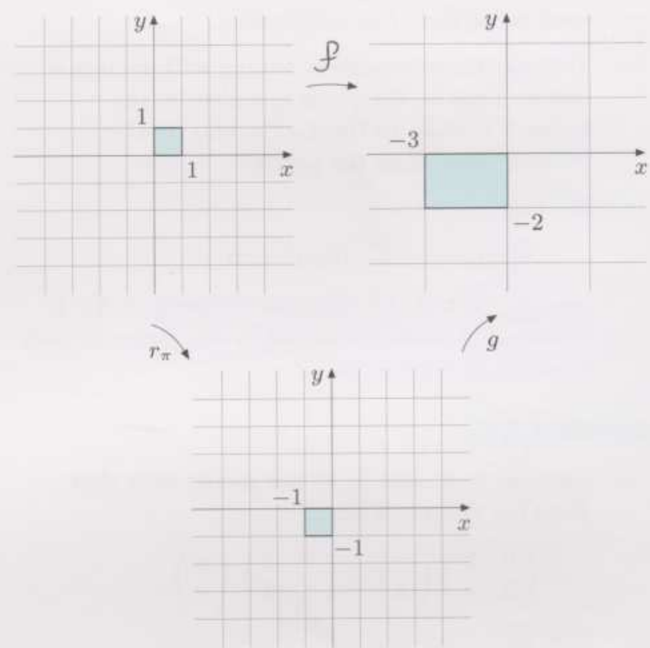


Figure S.19

Solution 3.6

Here

$$\det\mathbf{A} = 5 \times 6 - 8 \times 4 = -2 \neq 0,$$

so $\mathbf{A}$ has inverse

$$\mathbf{A}^{-1} = \frac{1}{-2}\begin{pmatrix}6 & -8\\ -4 & 5\end{pmatrix} = \begin{pmatrix}-3 & 4\\ 2 & -\frac{5}{2}\end{pmatrix}.$$

Thus

$$\mathbf{AA}^{-1} = \begin{pmatrix}5 & 8\\ 4 & 6\end{pmatrix}\begin{pmatrix}-3 & 4\\ 2 & -\frac{5}{2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{I},$$

and

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix}-3 & 4\\ 2 & -\frac{5}{2}\end{pmatrix}\begin{pmatrix}5 & 8\\ 4 & 6\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{I}.$$

Solution 3.7

(a) In this case,

$$\det \mathbf{A} = 3 \times 5 - 2 \times 9 = -3.$$

Since this determinant is non-zero, $\mathbf{A}$ is invertible. It follows that f is one-one and onto. Moreover, the inverse f^{-1} is the linear transformation represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} 5 & -2 \\ -9 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix}.$$

(b) The image, $f(S)$, of the unit square has vertices at the points with position vectors

$$\begin{pmatrix} 3 & 2 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 2 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 2 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 2 \\ 9 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

That is, $f(S)$ is the parallelogram with vertices $(0,0)$, $(3,9)$, $(5,14)$, $(2,5)$.

Under f^{-1}, the image $f(S)$ maps back to the figure with vertices at the points given by

$$\begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -5+6 \\ 9-9 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 14 \end{pmatrix} = \begin{pmatrix} -\frac{25}{3}+\frac{28}{3} \\ 15-14 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{5}{3} & \frac{2}{3} \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{10}{3}+\frac{10}{3} \\ 6-5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These vertices specify S.

(c) Under f, areas are scaled by the modulus of $\det \mathbf{A} = -3$. Under f^{-1}, areas are scaled by the modulus of

$$\det(\mathbf{A}^{-1}) = (-\tfrac{5}{3}) \times (-1) - (\tfrac{2}{3}) \times 3 = -\tfrac{1}{3}.$$

Thus f scales areas by the factor 3, and f^{-1} scales areas by the factor $\frac{1}{3}$. Each factor is the reciprocal of the other.

Solution 3.8

First observe that $\det \mathbf{A} = 2 \times 3 - 1 \times 1 = 5$, so f has an inverse transformation f^{-1} represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

If P is an arbitrary point (x,y) on the image $f(\mathscr{C})$, then P must be the image under f of the point $f^{-1}(P)$ on $\mathscr{C}$ with position vector

$$\mathbf{A}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{5}x - \frac{1}{5}y \\ -\frac{1}{5}x + \frac{2}{5}y \end{pmatrix}.$$

Since these components are the coordinates of a point on $\mathscr{C}$, it follows that

$$(\tfrac{3}{5}x - \tfrac{1}{5}y)^2 + (-\tfrac{1}{5}x + \tfrac{2}{5}y)^2 = 1,$$

or, equivalently,

$$\tfrac{1}{25}(9x^2 - 6xy + y^2 + x^2 - 4xy + 4y^2) = 1.$$

That is

$$\tfrac{1}{25}(10x^2 - 10xy + 5y^2) = 1,$$

or, equivalently,

$$2x^2 - 2xy + y^2 = 5.$$

This is therefore the equation of $f(\mathscr{C})$.

The area enclosed by $\mathscr{C}$ is π and f scales areas by the factor $|\det \mathbf{A}| = 5$, so the area enclosed by $f(\mathscr{C})$ is 5π.

Solution 3.9

(a) The ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ is centred at the origin and its vertices are the points $(3,0)$, $(-3,0)$, $(0,2)$ and $(0,-2)$. It follows that the unit circle can be mapped onto the ellipse by a scaling by the factor 3 parallel to the x-axis and by the factor 2 parallel to the y-axis. This scaling is the linear transformation f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix};$$

it scales areas by the factor $|\det \mathbf{A}| = 6$.

Since the unit circle has area π, it follows that the ellipse has area 6π.

(b) In this case, the unit circle is mapped onto the ellipse by applying the linear transformation represented by

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This transformation scales areas by the factor $|\det \mathbf{A}| = ab$, so the area of the ellipse is πab.

Solution 4.1

Here A has position vector $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$; also $\overrightarrow{AB}$ is

$$\begin{pmatrix} 7-5 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and $\overrightarrow{AC}$ is

$$\begin{pmatrix} 8-5 \\ 6-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

So the required transformation is

$$f(\mathbf{x}) = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Solution 4.2

(a) The required affine transformation f has the rule $\mathbf{x} \longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}$, where

$$\mathbf{A} = \begin{pmatrix} -1-4 & 3-4 \\ 5+3 & 7+3 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ 8 & 10 \end{pmatrix}$$

and

$$\mathbf{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

(b) It follows that the affine transformation scales areas by the factor

$$|\det \mathbf{A}| = |-5 \times 10 - (-1) \times 8| = 42.$$

The triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ has area $\frac{1}{2}$, so the area of T must be $42 \times \frac{1}{2} = 21$.

Solution 4.3

The rotation is given by the composite transformation $t_{3,1} \circ (r_{\pi/3} \circ t_{-3,-1})$, where $r_{\pi/3}$ is represented by the matrix

$$\mathbf{R}_{\pi/3} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

(Taking the common factor $\frac{1}{2}$ outside the matrix brackets makes the ensuing manipulations less clumsy.)

Under this composite transformation, an arbitrary point $\mathbf{x}$ is mapped to

$$\begin{aligned}
&\mathbf{R}_{\pi/3}\left(\mathbf{x} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\mathbf{x} - \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\mathbf{x} - \frac{1}{2}\begin{pmatrix} 3-\sqrt{3} \\ 3\sqrt{3}+1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\mathbf{x} + \frac{1}{2}\begin{pmatrix} 3+\sqrt{3} \\ 1-3\sqrt{3} \end{pmatrix}.
\end{aligned}$$

So this rotation about the point $(3,1)$ can be expressed as the affine transformation $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\mathbf{x} + \frac{1}{2}\begin{pmatrix} 3+\sqrt{3} \\ 1-3\sqrt{3} \end{pmatrix}.$$

(You should check that the centre of rotation, $(3,1)$, remains fixed under f.)

Solution 4.4

(a) Since the line $y = x$ makes an angle $\pi/4$ with the positive x-axis, it follows that reflection in this line is $q_{\pi/4}$; so the required matrix is

$$\mathbf{Q}_{\pi/4} = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ \sin(\pi/2) & -\cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(Alternatively, notice that the images of $(1,0)$ and $(0,1)$ under the reflection are $(0,1)$ and $(1,0)$, respectively. The position vectors of these images form the columns of the matrix.)

(b) The idea is to shift the plane 4 units down so that the line of reflection passes through the origin. After performing the reflection in the shifted line, the plane is then shifted back up 4 units so that the line of reflection ends up where it started. The required reflection is therefore $t_{0,4} \circ (q_{\pi/4} \circ t_{0,-4})$. Under this composite transformation, an arbitrary point $\mathbf{x}$ is mapped to

$$\begin{aligned}
&\mathbf{Q}_{\pi/4}\left(\mathbf{x} + \begin{pmatrix} 0 \\ -4 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 0 \\ 4 \end{pmatrix}\right) + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -4 \\ 4 \end{pmatrix}.
\end{aligned}$$

So reflection in the line $y = x + 4$ can be expressed as the affine transformation $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

(Note that a check on the working would be to show that an arbitrary point $(x, x+4)$ on the line of reflection remains fixed under f.)

Solutions to Exercises

Solution 1.1

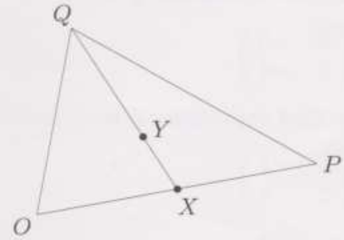


Figure S.20

We have

$$\mathbf{y} = \overrightarrow{OY} = \overrightarrow{OQ} + \overrightarrow{QY} = \mathbf{q} + \tfrac{2}{3}\overrightarrow{QX}.$$

But $\overrightarrow{QX} = \overrightarrow{QO} + \overrightarrow{OX} = -\mathbf{q} + \frac{1}{2}\mathbf{p}$, so

$$\mathbf{y} = \mathbf{q} + \tfrac{2}{3}(-\mathbf{q} + \tfrac{1}{2}\mathbf{p}) = \tfrac{1}{3}\mathbf{p} + \tfrac{1}{3}\mathbf{q}.$$

Solution 1.2

The reflection is represented by the matrix

$$\mathbf{Q}_{\pi/6} = \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}.$$

The image of the triangle therefore has vertices given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

So the image is the triangle with vertices at $(2, 0)$, $(0, 2)$ and $(0, -2)$.

Solution 1.3

(a) This matrix has the form

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where

$$\cos\theta = \tfrac{1}{2} \quad \text{and} \quad \sin\theta = \frac{\sqrt{3}}{2}.$$

These equations are satisfied by $\theta = \pi/3$.

(b) This matrix has the form

$$\mathbf{Q}_\theta = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

where

$$\cos(2\theta) = \tfrac{1}{2} \quad \text{and} \quad \sin(2\theta) = \frac{\sqrt{3}}{2}.$$

These equations are satisfied by $2\theta = \pi/3$, that is $\theta = \pi/6$. (You used the matrix $\mathbf{Q}_{\pi/6}$ in Exercise 1.2.)

Solution 2.1

(a) This matrix represents a scaling with factors 4 and 2. That is, the plane is scaled by the factor 4 parallel to the x-axis, and by the factor 2 parallel to the y-axis.

We have

$$\det \mathbf{A} = 4 \times 2 - 0 \times 0 = 8.$$

It follows that areas are scaled by the factor 8 and orientation is preserved.

(b) This matrix has the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with $\cos\theta = 0$ and $\sin\theta = -1$. These equations are satisfied by the value $\theta = \frac{3}{2}\pi$, for example. So f is the anticlockwise rotation $r_{3\pi/2}$.

We have

$$\det \mathbf{A} = 0 \times 0 - 1 \times (-1) = 1.$$

It follows that areas and orientation are preserved.

(c) The first column of this matrix is $\frac{4}{3}$ of the second column, so this matrix represents a flattening. Indeed, the image of an arbitrary point (x, y) has position vector

$$x\begin{pmatrix} 8 \\ 4 \end{pmatrix} + y\begin{pmatrix} 6 \\ 3 \end{pmatrix} = (4x + 3y)\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

so f is a flattening onto the line $x - 2y = 0$.

We have

$$\det \mathbf{A} = 8 \times 3 - 6 \times 4 = 0.$$

This is consistent with the fact that a flattening scales areas to 0. Orientation is destroyed.

Solution 2.2

Since f sends $(1,0)$ to $(-1,4)$ and $(0,1)$ to $(2,3)$, it is represented by the matrix

$$\begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix}.$$

This matrix has determinant $-1 \times 3 - 2 \times 4 = -11$, so f scales areas by a factor of 11. The triangle T is the image of the right-angled triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$. Since this right-angled triangle has area $\frac{1}{2}$, it follows that T has area $5\frac{1}{2}$.

Solution 2.3

(a) $$\begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}$$

(b) $$\begin{pmatrix} \cos(-\pi/6) & -\sin(-\pi/6) \\ \sin(-\pi/6) & \cos(-\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

(c) $$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

(d) $$\begin{pmatrix} 2 & -5 \\ 3 & 4 \end{pmatrix}$$

Solution 3.1

(a) The composite $g \circ f$ is the linear transformation represented by the matrix

$$\mathbf{BA} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 10 \\ 21 & 17 \end{pmatrix}.$$

(b) The composite $f \circ g$ is the linear transformation represented by the matrix

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 18 & 29 \end{pmatrix}.$$

Solution 3.2

(a) In this case

$$\det \mathbf{A} = 1 \times 4 - 2 \times 3 = -2.$$

Since this determinant is non-zero, it follows that $\mathbf{A}$ is invertible. Moreover,

$$\mathbf{A}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

(b) Here

$$\det \mathbf{A} = 0 \times 4 - 0 \times 3 = 0.$$

Since this determinant is zero, it follows that $\mathbf{A}$ is not invertible.

(c) In this case

$$\det \mathbf{A} = 2 \times 4 - (-1) \times (-8) = 0.$$

Since this determinant is zero, it follows that $\mathbf{A}$ is not invertible.

(d) Here

$$\det \mathbf{A} = 2 \times 4 - 3 \times 3 = -1.$$

Since this determinant is non-zero, it follows that $\mathbf{A}$ is invertible. Moreover,

$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}.$$

Solution 3.3

(a) The matrix that represents f is

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}.$$

The matrix that represents g is

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

(b) First observe that

$$\det \mathbf{A} = 2 \times (-4) - (-3) \times 3 = 1,$$

so f has an inverse transformation f^{-1}. The required linear transformation is therefore f^{-1}, which is represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix}.$$

(c) We know that f^{-1} sends $(2,3)$ and $(-3,-4)$ to $(1,0)$ and $(0,1)$, respectively. We also know that g sends $(1,0)$ and $(0,1)$ to $(1,2)$ and $(-1,1)$, respectively. The required linear transformation is therefore the composite $g \circ f^{-1}$. This is represented by the matrix

$$\begin{aligned}\mathbf{BA}^{-1} &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ -11 & 8 \end{pmatrix}.\end{aligned}$$

As a check, notice that

$$\begin{pmatrix} -1 & 1 \\ -11 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 1 \\ -11 & 8 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

as required.

Solution 3.4

First observe that $\det \mathbf{A} = 2 \times 4 - 3 \times 2 = 2$, so f has an inverse transformation f^{-1} represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{3}{2} \\ -1 & 1 \end{pmatrix}.$$

If P is an arbitrary point (x,y) on the image $f(\mathscr{C})$, then P must be the image under f of the point $f^{-1}(P)$ on $\mathscr{C}$ with position vector

$$\mathbf{A}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -\frac{3}{2} \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - \frac{3}{2}y \\ -x + y \end{pmatrix}.$$

Since these components are the coordinates of a point on $\mathscr{C}$, it follows that

$$(2x - \tfrac{3}{2}y)^2 + (-x+y)^2 = 1,$$

or, equivalently,

$$4x^2 - 6xy + \tfrac{9}{4}y^2 + x^2 - 2xy + y^2 = 1.$$

That is

$$5x^2 - 8xy + \tfrac{13}{4}y^2 = 1,$$

or, equivalently,

$$20x^2 - 32xy + 13y^2 = 4.$$

This is therefore the equation of $f(\mathscr{C})$.

The area enclosed by $\mathscr{C}$ is π and f scales areas by the factor $|\det \mathbf{A}| = 2$, so the area of $f(\mathscr{C})$ is 2π.

Solution 4.1

(a) The required affine transformation f has the rule $\mathbf{x} \longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}$, where

$$\mathbf{A} = \begin{pmatrix} 10-4 & 8-4 \\ 5-(-2) & 7-(-2) \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 7 & 9 \end{pmatrix}$$

and

$$\mathbf{a} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

(b) It follows that the affine transformation scales areas by the factor

$$|\det \mathbf{A}| = |6 \times 9 - 4 \times 7| = 26.$$

Since the triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ has area $\frac{1}{2}$, the area of T must be $26 \times \frac{1}{2} = 13$.

Solution 4.2

The rotation is given by the composite transformation $t_{4,6} \circ (r_{\pi/4} \circ t_{-4,-6})$. Now

$$\mathbf{R}_{\pi/4} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Thus, under this composite transformation, an arbitrary point $\mathbf{x}$ is mapped to

$$\begin{aligned}
&\mathbf{R}_{\pi/4}\left(\mathbf{x} + \begin{pmatrix} -4 \\ -6 \end{pmatrix}\right) + \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 4 \\ 6 \end{pmatrix}\right) + \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x} - \frac{1}{\sqrt{2}}\begin{pmatrix} -2 \\ 10 \end{pmatrix} + \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4+\sqrt{2} \\ 6-5\sqrt{2} \end{pmatrix}.
\end{aligned}$$

So this rotation about the point $(4,6)$ can be expressed as the affine transformation $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4+\sqrt{2} \\ 6-5\sqrt{2} \end{pmatrix}.$$

Index

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